GEOMETRY OF TEICHMÜLLER SPACES

by

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ABSTRACT

In this thesis, we present the geometric theory of Teichmüller spaces. The quotient of the Teichmüller space by the Teichmüller modular group can be identified with Riemann's moduli space of all complex structures on an oriented two-dimensional differentiable manifold. We study various ways of constructing Teichmüller spaces and suitably identify them. The most important analytic tool employed by Teichmüller, the quasiconformal mappings, is considered in detail. Using the Fuchsian model and the induced hyperbolic geometry on a Riemann surface, we construct the Fricke coordinates and the Fenchel-Nielsen coordinates on Teichmüller spaces. Then we follow Ahlfors and Bers to show that the Teichmüller space T_g for genus $g \geq 2$ has a natural complex manifold structure of dimension $3g-3$, thus verifying Riemann's count of the number of parameters for the moduli space M_g. The action of the Teichmüller modular group being discrete, the moduli space has a natural structure of a normal analytic space. We present the Teichmüller's distance and the Weil-Petersson's metric on Teichmüller spaces, the geometry of which finds significant applications in various branches of Mathematics in recent years.

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CONTENTS

CHAPTER 0

Introduction

In this thesis, we present the geometric theory of Teichmüller spaces which give a parametrization of all the complex structures on a given surface.

Riemann's moduli problem asks how many complex structures there are on a given oriented two dimensional differentiable manifold which are not biholomorphically equivalent. The set M_g of all biholomorphic equivalence classes of closed Riemann surface of genus g is called the Riemann's moduli space of genus g. One observes that M_0 has a single element and by the theory of elliptic functions that M_1 is the complex plane. In 1857, Riemann asserted that M_g, $g \geq 2$, is parametrized by 3g-3 complex parameters. His argument is by representing closed Riemann surfaces of genus g as finite branched coverings of the Riemann surface and counting the number of degrees of freedom of the branch points.

It turns out that M_g can be given a normal complex analytic space structure. This can be shown by constructing the Teichmüller space T_g of all marked closed Riemann surfaces of genus g and taking quotient by the Teichmüller modular group Mod_g which corresponds to the change of markings. Then $M_g \cong T_g/Mod_g$ is a normal complex analytic space by Cartan's Theorem. [Ba]

The Teichmüller space T_g already appeared though implicitly in the continuity argument of Klein and Poincaré in their study of Fuchsian groups and automorphic functions in the 1880's. Then Fricke, Fenchel and Nielsen constructed T_g, $g \geq 2$, as a real (6g-6)-dimensional manifold, using the representation of any closed Riemann surface of genus g as the quotient H/Γ of the upper half plane H by a Fuchsian group Γ. Around 1940, Teichmüller discovered an important relation between extermal quasiconformal mappings and holomorphic quadratic differentials. In fact the use of quasiconformal mappings in the moduli problem has been one of his great innovations. He was then able to assert that T_g is homeomorphic to $\mathbf{R}^{6g-6}$ and to introduce the Teichmüller distance on T_g.

In the end of the 1950's, Ahlfors and Bers gave rigorous proofs for Teichmüller's results. They also showed that T_g, $g \geq 2$, has a natural complex structure of dimension 3g-3, by embedding T_g as a bounded domain in the complex vector space $A_2(R)$ of holomorphic quadratic differentials of a closed Riemann surface R of genus g. With respect to this complex structure, the Teichmüller modular group Mod_g turns out to be a discrete group of biholomorphic automorphism of T_g.

Weil introduced a Kähler metric on the Teichmüller space T_g based on Petersson's Hermitian pairing for automorphic forms. The metric is invariant under the covering of T_g onto M_g. Ahlfors considered the differential geometry of this metric. He found that the Ricci, holomorphic sectional and scalar curvatures are all negative. Tromba showed that the sectional curvature is also negative. Unfortunately, Chu and Wolpert showed that the metric is not complete. [Ch] [Wo 1] Important works on the geometry of the moduli space can be found in [Wo 3 - Wo 6].

Recently the theory of Teichmüller spaces has found significant applications in many areas of mathematics, including automorphic forms, complex analysis, algebraic geometry, differential geometry, topology in three dimension and complex dynamics, as well as in string theory in physics.

In Chapter 1, we discuss two ways of constructing Teichmüller spaces. The first one is by considering marked Riemann surfaces of genus g. The second one is by considering orientation-preserving diffeomorphisms from a fixed

Riemann surface R of genus g. The resulting Teichmüller spaces, T_g and T(R) respectively, can be identified and their elements represent deformations of the complex structure on R. Further, they can be identified with a quotient space of the Beltrami coefficients, which is the next object we consider. We then introduce the Fuchsian model of R and present some basic results. Finally we construct the Fricke space which represents T_g as a subset of $\mathbf{R}^{6g-6}$.

In Chapter 2, we discuss the Fenchel-Nielsen coordinates on T_g using the hyperbolic geometry induced by the Poincaré metric. The method is to decompose Riemann surfaces into 2g-2 pairs of pants by 3g-3 simple closed geodesics. Then we define the Fenchel-Nielsen coordinates on T_g by means of geodesics length functions of the simple closed geodesics and twist parameters along these geodesics. We also present the Fricke-Klein embedding.

In Chapter 3, we study quasiconformal mappings. Two analytic and one geometric definitions of quasiconformal mappings will be given. The existence and uniqueness of a quasiconformal mapping satisfying a given Beltrami equation and the dependence of solutions on Beltrami coefficients will be discussed.

In Chapter 4, Teichmüller spaces are constructed again by using quasiconformal mappings instead. The Teichmüller distance is then defined and its completeness can be proved. Applying the Teichmüller existence and uniqueness theorems, we can show that the Teichmüller space is homeomorphic to the space of holomorphic quadratic differentials $A_2(R)_1$ on R. Hence Teichmüller space is homeomorphic to $\mathbf{R}^{6g-6}$ by the Riemann-Roch theorem.

In Chapter 5, we construct the Bers' embedding of T(R) into a bounded domain in $A_2(R^*)$, R^* = the mirror image of R, by using the Schwarzian derivatives. Via the embedding, T(R) has a natural complex manifold structure of dimension 3g-3. It is then shown that the Teichmüller modular group acts properly discontinuously as a subgroup of biholomorphic automorphism proving the moduli space with a normal complex analytic space structure. Finally, we discuss on the classification of Teichmüller modular transformations by means of Bers' extremal problems.

In Chapter 6, we introduce the Weil-Petersson metric on T_g and discuss its basic properties including its Kählerity. The holomorphic tangent space of T_g at the base point is identified with the dual space of $A_2(R)$. The holomorphic tangent space elsewhere is computed by means of harmonic Beltrami differentials.

CHAPTER 1

Teichmüller Space of genus g

1. Teichmüller Space of genus g

The Teichmüller space of genus g is constructed in two ways.

The first construction is given by considering marked Riemann surfaces. Let R be a closed Riemann surface of genus g and $p \in R$. A system of canonical generators $\Sigma_p = \{[A_j],[B_j]\}_{j=1}^{g}$ of $\pi_1(R,p)$ is called a *marking* on R.

Let $\Sigma_p = \{[A_j],[B_j]\}_{j=1}^{g}$ and $\Sigma_q = \{[A'_j], [B'_j]\}_{j=1}^{g}$ be markings on Riemann surfaces R and S of genus g. (R, Σ_p) and (S, Σ_q) are said to be *equivalent* if there exists a biholomorphic mapping $h : S \to R$ and a continuous curve C_0 on R such that $h_*[A'_j] = [C_o^{-1} \cdot A_j \cdot C_0]$ and $h_*[B'_j] = [C_o^{-1} \cdot B_j \cdot C_0]$. The equivalence class $[R, \Sigma_p]$ of (R, Σ_p) is called *a marked closed Riemann surface of genus g*. The *Teichmüller space T_g of genus g* is the set of all marked closed Riemann surfaces of genus g.

The second construction is given by considering orientation-preserving diffeomorphisms. Fix a Riemann surface R of genus g. Let $f : R \to S$ be an orientation-preserving diffeomorphism onto a Riemann surface S. Two pairs (S, f) and (T, g) are said to be *equivalent* if $g \circ f^{-1} : S \to T$ is homotopic to a biholomorphic mapping $h : S \to T$. The *Teichmüller space* T(R) of R is the set of all the equivalence classes [S, f] of (S, f).

Let us consider the relationship between T_g and T(R). Fix a marking $\Sigma = \{[A_j],[B_j]\}_{j=1}^{g}$ on R with base point p. Corresponding to [S, f] in T(R), a marking $f_*(\Sigma)$ on S determines $[S, f_*(\Sigma)]$ in T_g. It is noted that $[S, f_*(\Sigma)]$ in T_g does not depend on a representative of [S, f] in T(R). One then defines a mapping $\phi_\Sigma : T(R) \to T_g$ by $\phi_\Sigma([S,f]) = [S, f_*(\Sigma)]$ and checks that ϕ_Σ is bijective. Thus $T_g \cong T(R)$.

Finally, we consider a canonical group action on T(R). The set of all homotopy classes [w] of orientation-preserving diffeomorphisms $w : R \to R$ is called *the Teichmüller modular group* Mod(R). Every [w] acts on T(R) by

$$[w]_*([S,f]) = [S, f \circ w^{-1}] \qquad \text{for all } [S,f] \text{ in } T(R).$$

$[w]_*$ is called *the Teichmüller modular transformation of* [w].

The set M_g all biholomorphic equivalence classes [S] of Riemann surface S of genus g is called *the moduli space of closed Riemann surface of genus g.* Since for an arbitrary closed Riemann surface S of genus g there exists an orientation-preserving diffeomorphism of R onto S, $M_g \cong T(R)/\mathrm{Mod}(R)$.

Beltrami Coefficients

For [S, f] in T(R), we want to compare the complex structures of R and S. Take coordinates neighborhoods (U, z) and (V, w) on R and S respectively with f(U) in V, and $F := w \circ f \circ z^{-1}$. Then *the Beltrami coefficient of f with respect to (U,z)* $\mu = \frac{F_{\bar{z}}}{F_z}$ is a smooth function defined in z(U). One observes that $|\mu| < 1$ by virtue the fact that F is an orientation-preserving diffeomorphism. It is independent of the choice of a local coordinate w on S but it depends on the choice of z on R. In fact, take coordinate neighborhoods (U_j, z_j) and (U_k, z_k) of R and (V_j, w_j) and (V_k, w_k) of S such that $f(U_j)$ in V_j and $f(U_k)$ in V_k. Let μ_j and μ_k be the Beltrami coefficients of f with respect to (U_j, z_j) and (U_k, z_k), respectively. Then

$$\mu_j = (\mu_k \circ z_{kj}) \, \overline{\frac{dz_{kj}}{dz_j}} \Big/ \frac{dz_{kj}}{dz_j} \qquad \text{on } z_j(U_j \cap U_k)$$

where $z_{kj} = z_k \circ z_j^{-1}$.

According to this transformation formula the set of Beltrami coefficient of f on coordinate neighborhoods of R induces a differential form of type (-1,1) $\mu(z)\frac{d\bar{z}}{dz}$ on R, called *the Beltrami coefficient of f on R.* Also $|\mu(z)|$ does not depend on the local coordinates on R. So $|\mu(z)|$ is a continuous function on R. The L^∞ - norm is given by $\|\mu\|_\infty = \sup_{z\in R} |\mu(z)| < 1$.

Let $f : R \to S$ be an orientation-preserving diffeomorphism and $\{(V_\alpha, w_\alpha)\}_{\alpha\in A}$ be a system of coordinate neighborhoods on S. Then the system of coordinate neighborhoods $\{(f^{-1}(V_\alpha), w_\alpha \circ f)\}_{\alpha\in A}$ defines a complex structure on R. The resulting Riemann surface is denoted by R_f. Thus $[S,f] = [R_f, id]$ in T(R) represents a deformation of the complex structure on R, and the Beltrami coefficient of f is used to measure the deviation of f from conformality.

Proposition 1.1. *For Riemann surfaces R, S and T and orientation-preserving diffeomorphisms $f : R \to S$ and $g : S \to T$. Then*

$$\mu_g \circ f = \frac{\overline{f_z}}{f_z} \frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu_f}\, \mu_{g\circ f}} .$$

In particular, for orientation-preserving diffeomorphisms $f_j: R \to S_j$ $(j=1,2)$, $f_2 \circ f_1^{-1} : S_1 \to S_2$ is biholomorphic $\Leftrightarrow$ $\mu_{f_1} = \mu_{f_2}$.

We consider the geometric meaning of the Beltrami coefficients. Let $f : D \to D'$ be an orientation-preserving diffeomorphism between domains D, D' in **C**. Let $L(z) = f_z(z_0)\, z + f_{\bar z}(z_0)\, \bar z$ be the linear term of the Taylor expansion of f at $z_0 \in D$. The linear mapping L sends a circle in the z-plane to an ellipse in the w-plane. Since

$$|f_z(z_0)|\ (1-|\mu_f(z_0)|)|z_0| \le |L(z)| \le |f_z(z_0)|\ (1+|\mu_f(z_0)|)|z_0| ,$$

the ratio of the major axis to the minor axis of the ellipse is

$$K_f(z_0) = \frac{1+|\mu_f(z_0)|}{1-|\mu_f(z_0)|},$$

where $\mu_f(z_0) = \dfrac{f_{\bar z}(z_0)}{f_z(z_0)}$ is the Beltrami coefficient of f at z_0. $f : D \to D'$ is called a *quasiconformal mapping* if $K_f = \sup_{z\in D} \dfrac{1+|\mu_f(z)|}{1-|\mu_f(z)|} < \infty$. K_f is called *the maximal dilatation of f*.

Let $B(R)_1$ be the set of Beltrami coefficients of all orientation-preserving diffeomorphisms of a fixed Riemann surface R on Riemann surfaces. Define a topology on $B(R)_1$ by using the L^∞ -norm.

Let $Diff_+(R)$ be the group of all orientation-preserving diffeomorphisms on R onto R and let $Diff_0(R)$ be a normal subgroup which consists of all elements in $Diff_+(R)$ homotopic to id.

The action of $Diff_+(R)$ on $B(R)_1$ is given by

$$w^*(\mu_f) = \mu_{f \circ w^{-1}} = \frac{\overline{w_z}}{w_z} \frac{\mu_f - \mu_w}{1 - \overline{\mu_w}\, \mu_f} \circ w^{-1}$$

where $w \in Diff_+(R)$ and $\mu_f \in B(R)_1$.

Theorem 1.2. *For orientation-preserving diffeomorphisms $f : R \to S$ and $g : R \to S'$, there exists a biholomorphic mapping $h : S \to S' \Leftrightarrow \mu_g = w^*(\mu_f)$ for some w in $Diff_+(R)$.*

Moreover, $g^{-1} \circ h \circ f$ is homotopic to identity of $R \Leftrightarrow w \in Diff_0(R)$.

Proof . $\Rightarrow$ $w := g^{-1} \circ h \circ f$. Then $\mu_g = \mu_{h \circ f \circ w^{-1}} = \mu_{f \circ w^{-1}} = w^*(\mu_f)$

$\Leftarrow$ By the above proposition, $h := g \circ w \circ f^{-1}$ is biholomorphic.

□

Corollary. *The mapping which sends (S,f) to μ_f in $B(R)_1$ induces*

$$T(R) \cong B(R)_1/Diff_0(R)$$

$$M_g \cong B(R)_1/Diff_+(R).$$

2. Fuchsian model and Discrete subgroups of Aut(H)

Definition. If a universal covering surface of a Riemann surface R is the upper half plane H, its covering transformation group Γ is called a *Fuchsian* *model* of R. An open set F of H is a *fundamental domain* for a Fuchsian model Γ if F satisfies

1. $\gamma(F) \cap F = \phi \qquad \forall \gamma \in \Gamma \backslash \{id\}$

2. $H = \bigcup_{\gamma \in \Gamma} \gamma(\overline{F})$ where $\overline{F}$ is the closure of F in H

3. The boundary ∂F of F in H has measure zero with respect to 2-dimensional Lebesgue measure.

We take the compact-open topology on Aut(H). That is, $\{\gamma_n\}_{n=1}^{\infty}$ of Aut(H) converges to $\gamma \in$ Aut(H) if γ_n converges uniformly to γ on compact subsets of H as $n \to \infty$. Identifying Aut(H) with the Lie group PSL(2,**R**), we observe this topology is equivalent to the Lie group topology.

A discrete subgroup of Aut(H) is called a *Fuchsian group.* Since SL(2,**R**) is second countable, a Fuchsian group consists of a countable number of elements.

Proposition 1.3. *For a subgroup Γ of Aut(H), the following conditions are equivalent:*

1. Γ is a Fuchsian group.

2. There exists no sequence of mutually distinct elements of Γ which converges in Aut(H).

3. Γ acts properly discontinuously on H.

Proof. (2)$\Rightarrow$(1) and (3)$\Rightarrow$(1) follow from the definition.

(1)$\Rightarrow$(2) If not. $\exists$ a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of distinct elements of Γ which converges to $\gamma \in$ Aut(H). Then $\{\gamma_n^{-1}\}_{n=1}^{\infty}$ converges to $\gamma^{-1} \in$ Aut(H). Since $\gamma_n^{-1} \circ \gamma_{n+1} \in \Gamma \backslash \{id\}$ and id is not an isolated point of Γ, Γ is not discrete.

(1)⇒ (3) If not. $\exists z_0 \in H$ and a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of distinct elements of Γ such that $\gamma_n(z_0) \to w_0 \in H$ as $n \to \infty$. Since $\{\gamma_n\}_{n=1}^{\infty}$ is a normal family, we assume that $\{\gamma_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of H to a holomorphic function γ on H. From the following lemma $\gamma \in Aut(H)$. Γ is not Fuchsian.

□

Lemma 1.4. *Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of Aut(H) which converges uniformly on compact subsets of H to a holomorphic function f on H. Then $f \in Aut(H)$ or f is a constant function $c \in \hat{\mathbf{R}}$.*

Proposition 1.5. *Let Γ is a Fuchsian model of a Riemann surface of genus $g \geq 2$. $\forall \zeta \in \hat{\mathbf{R}}$, $\exists$ a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of Γ such that $\{\gamma_n(z)\}_{n=1}^{\infty}$ converges to ζ $\forall z \in H$.*

Lemma 1.6. *Let γ be a hyperbolic element of a Fuchsian group Γ and $\delta \in \Gamma\backslash\{id\}$. Then the fixed point sets satisfy either $Fix(\gamma) = Fix(\delta)$ or $Fix(\gamma) \cap Fix(\delta) = \phi$.*

Proof. If not. By Aut(H)-conjugation, assume that $Fix(\gamma) = \{0,\infty\}$ and $Fix(\gamma) \cap Fix(\delta) = \{\infty\}$. Then matrix representations $A = \begin{bmatrix} \lambda & o \\ o & \lambda^{-1} \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ o & a^{-1} \end{bmatrix}$ of γ and δ respectively, where $\lambda \in \mathbf{R}^+\backslash\{1\}$ and $a,b \in \mathbf{R}\backslash\{0\}$. $C_n = B A^n B^{-1} A^{-n} = \begin{bmatrix} 1 & ab(1-\lambda)^{2n} \\ 0 & 1 \end{bmatrix}$ for all $n \in \mathbf{Z}$.

If $0 < \lambda < 1$, then $C_n \to C := \begin{bmatrix} 1 & ab \\ 0 & 1 \end{bmatrix}$ as $n \to +\infty$. If $\lambda > 1$, then $C_n \to C := \begin{bmatrix} 1 & ab \\ 0 & 1 \end{bmatrix}$ as $n \to -\infty$. Thus $\{ C_n \}_{n=-\infty}^{\infty}$ consists of distinct elements. By the previous theorem, Γ is not discrete.

□

Lemma 1.7. *Let Γ be a Fuchsian group containing $\gamma_0(z) = z + 1$. Then every $\gamma \in \Gamma$ having a real matrix representation $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $|A| = 1$ satisfies $|c| \geq$*

1 provided that $c \neq 0$.

Proof. If not. There exists $\gamma \in \Gamma$ such that γ has a matrix representation $A \in SL(2,\mathbf{R})$ with $0 < |c| < 1$. Let $A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be a matrix representation of γ_0. $A_1 = A$ and $A_{n+1} = A_n A_0 A_n^{-1}$. for all $n \in \mathbf{N}$ where

$$A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} := \begin{bmatrix} 1-a_{n-1}c_{n-1} & a_{n-1}^2 \\ -c_{n-1}^2 & 1+a_{n-1}c_{n-1} \end{bmatrix}.$$

Thus $c_n = -c^{2^{n-1}} \to 0$ as $n \to \infty$. Inductively, $|a_n| \leq \max \{ |a| , \frac{1}{1-|c|} \}$ $\forall\ n \in \mathbf{N}$.

So $A_n \to A_0$ as $n \to \infty$. It contradicts the discreteness of Γ.

□

Theorem 1.8. *Every element of a Fuchsian model of a Riemann surface of genus $g \geq 2$ consists of the identity and hyperbolic elements.*

Proof . Since every $\gamma \in \Gamma\backslash\{id\}$ has no fixed points on H, it is hyperbolic or parabolic. Assume that Γ contains a parabolic element γ_0. By Aut(H)-conjugation, $\gamma_0(z) = z + 1$. From lemma 1.6, any $\gamma \in \Gamma\backslash\{id\}$ with $\gamma(\infty) = \infty$ is parabolic, which is written as $\gamma(z) = z + b$ for $b \in \mathbf{R}$. Hence $\Gamma_\infty = \{ \gamma \in \Gamma : \gamma(\infty) = \infty \}$ is a cyclic group. γ_0 is supposed to be a generator for Γ_∞.

Since every $\gamma(z) = \frac{az+b}{cz+d}$ with $ad-bc = 1$, $\gamma \in \Gamma\backslash\Gamma_\infty$ satisfies $c \neq 0$. So $|c| \geq 1$. Thus $\mathrm{Im}\gamma(z) \leq \frac{1}{(\mathrm{Im}z)|c|^2} \leq 1$ for all $\mathrm{Im}z > 1$. $U_0 := \{ z \in H : \mathrm{Im} > 2 \}$. Then any 2 points on U_0 are not equivalent under any element of $\Gamma\backslash\Gamma_\infty$. So $D_0 := U_0/\Gamma_\infty$ is biholomorphic to a domain R_0 in R and to $\{ z \in \mathbf{C} : 0 < |z| < 1 \}$. Because γ_0 corresponds to a non-trivial element of the fundamental group of R, $\overline{R_0}$ is not simply connected. Thus $\overline{R_0}$ is homeomorphic to $\{ z \in \mathbf{C} : 0 < |z| \leq 1 \}$. This contradicts that R is compact.

□

3. Fricke Space

Let Γ be a Fuchsian model of a Riemann surface R of genus $g \geq 2$. $\forall\ \delta \in$

Aut(H), $\Gamma' = \delta\,\Gamma\,\delta^{-1}$ is also a Fuchsian model of R. In order to assign uniquely a Fuchsian model Γ to a given marking $\Sigma = \{ [A_j] , [B_j] \}_{j=1}^{g}$ on R, we impose the normalization conditions :

1. β_g has its repelling and attractive fixed points at 0 and ∞ respectively.

2. α_g has its attractive fixed point at 1.

where α_j and β_j are the elements of Γ correspond to $[A_j]$ and $[B_j]$ respectively.

Proposition 1.9. *For a given marking Σ on a Riemann surface R of genus g ≥ 2, a canonical system of generators $\{ \alpha_j , \beta_j \}_{j=1}^{g}$ of a Fuchsian model Γ of R which satisfies the normalization conditions with respect to Σ is uniquely determined by $[R , \Sigma]$ in T_g.*

Proof. Let R' be another Riemann surface of genus g and S' be a marking on it such that $[R , \Sigma] = [R' , \Sigma']$ in T_g. Then there exists a biholomorphic mapping $f : R \to R'$ such that $f_*(\Sigma)$ is equivalent to Σ'. A lift $\tilde{f} \in$ Aut(H) of f is taken to satisfy $\alpha'_j = \tilde{f} \circ \alpha_j \circ \tilde{f}^{-1}$ and $\beta'_j = \tilde{f} \circ \beta_j \circ \tilde{f}^{-1}$, where $\{ \alpha'_j , \beta'_j \}_{j=1}^{g}$ is the canonical system of generators of a Fuchsian model of R' which satisfies the normalization conditions with respect to Σ'. From (1) and (2), $\tilde{f} = \mathrm{id}$. Thus $\alpha_j = \alpha'_j$ and $\beta_j = \beta'_j$.

□

Lemma 1.10. *Let $\{ \alpha_j , \beta_j \}_{j=1}^{g}$ is the canonical system of generators of the normalized Fuchsian model Γ for $[R,\Sigma]$ in T_g. If an element $\gamma(z) = \frac{az+b}{cz+d}$ of $\{ \alpha_j , \beta_j \}_{j=1}^{g}$ does not coincide with β_g, then $bc \neq 0$.*

By this lemma, the canonical system $\{ \alpha_j , \beta_j \}_{j=1}^{g}$ of generators of the normalized Fuchsian model Γ for $[R,\Sigma]$ in T_g is uniquely written in the form

$$\alpha_j = \frac{a_j z+b_j}{c_j z+d_j} \qquad \beta_j = \frac{a'_j z+b'_j}{c'_j z+d'_j}$$

where $a_j, b_j, a'_j, b'_j \in \mathbf{R}$, $c_j, c'_j > 0$ and $a_j d_j - b_j c_j = a'_j d'_j - b'_j c'_j = 1$ $(j = 1,\dots g-1)$.

The *Fricke coordinates* $\mathcal{F}_g : T_g \to \mathbf{R}^{6g-6}$ are given by

$$\mathcal{F}_g([R,\Sigma]) = (a_1,c_1,d_1,a'_1,c'_1,d'_1,\ldots a_{g-1},c_{g-1},d_{g-1},a'_{g-1},c'_{g-1},d'_{g-1}).$$

It can be checked that $\mathcal{F}_g$ is injective. The image $F_g = \mathcal{F}_g(T_g)$ is called the *Fricke space* of Riemann surface of genus g. We take the relative topology of F_g in $\mathbf{R}^{6g-6}$. Since $\mathcal{F}_g : T_g \to F_g$ is bijective, a topology on T_g is defined by identifying T_g with F_g.

CHAPTER 2

Hyperbolic Geometry and Fenchel-Nielsen Coordinates

1. Poincaré Metric and Hyperbolic Geometry

Poincaré Metric

The Poincaré metric on H is defined by $ds_H^2 = \frac{|dz|^2}{(\mathrm{Im} z)^2}$. The Möbius transformation $\gamma : H \to \Delta$ given by $\gamma(z) = \frac{z-i}{z+i}$ induces the Poincaré metric $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$ on Δ.

The *Poincaré distance* function on Δ is defined as follows. $\forall\, z_1, z_2 \in \Delta$,

$\rho(z_1,z_2) = \inf_C \int_C \frac{2|dz|}{1-|z|^2}$ where C varies over all rectifiable curves in Δ which connect z_1 and z_2.

Schwarz-Pick's lemma 2.1. *Every holomorphic mapping $f : \Delta \to \Delta$ satisfies* $\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}$.

If the equality holds at one point in Δ, then f is a biholomorphic automorphism, and the equality holds at all points.

Corollary. *Every holomorphic mapping $f : \Delta \to \Delta$ satisfies*

$$\rho(f(z_1),f(z_2)) \le \rho(z_1,z_2) \qquad z_1, z_2 \in \Delta.$$

Geodesics

For every rectifiable closed arc C in Δ, $l(C) := \int_C ds$ is called the *hyperbolic length* of C. $\forall\, z_1, z_2 \in \Delta$, a rectifiable closed arc C is called a *geodesic* connecting z_1 and z_2 in Δ if $\rho(z_1,z_2) = l(C)$.

Proposition 2.2. *$\forall z_1, z_2 \in \Delta$, $\exists$ a unique geodesic connecting z_1 and z_2 in Δ. Moreover, it is a subarc of the circle or the line segment which passes through z_1 and z_2 and is orthogonal to $\partial\Delta$.*

Remark. When $\gamma \in \mathrm{Aut}(\Delta)$ is hyperbolic, γ has 2 distinct fixed points on $\partial\Delta$. The part in Δ of the circle or the line segment which passes through these points and is orthogonal to $\partial\Delta$ is called the *axis of A_γ* of γ. A_γ is invariant under the action by γ.

Hyperbolic Metric on a Riemann Surface

Let R be a Riemann surface whose universal covering is biholomorphically equivalent to Δ. The covering transformation group Γ acting on Δ will be called a Fuchsian model of R. Let $\pi : \Delta \to R = \Delta/\Gamma$ be the projection. Since the Poincaré metric ds^2 is invariant under the action by Γ, a Riemannian metric ds_R^2 on R which satisfies $\pi^*(ds_R^2) = ds^2$ is obtained and is called the *hyperbolic metric* on R. Since $\Gamma \cong \pi_1(R,p_0)$, $\gamma \in \Gamma$ corresponds to $[C_\gamma] \in \pi_1(R,p_0)$. In particular, γ determines the free homotopy class of C_γ. We say that *γ covers the closed curve C_γ.*

When $\gamma \in \Gamma$ is hyperbolic, the closed curve $L_\gamma = A_\gamma/\langle\gamma\rangle$, the image on R of A_γ, is the unique geodesic belonging to the free homotopy class of C_γ. L_γ is called the *closed geodesic corresponding to γ.*

Proposition 2.3. *Let R be a Riemann surface with universal covering H, and Γ be a Fuchsian model of R acting on H. Let*

$$\gamma(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbf{R}, \quad ad-bc = 1,$$

be a hyperbolic element of Γ, and L_γ be the closed geodesic on R corresponding to γ. Then $tr^2(\gamma) = 4\cosh^2\left(\frac{l(L_\gamma)}{2}\right)$.

Pants

Consider a Riemann surface R which admits the hyperbolic metric cut by a family of mutually disjoint simple closed geodesics on R. Let P be a relative compact connected component of the resulting union of subsurfaces. If P contains no more simple closed geodesic of R, P is triply connected and homeomorphic to $\{ |z| < 2\} \setminus \{ |z+1| < \frac{1}{2}$ or $|z-1| < \frac{1}{2} \}$. A relative compact subsurface P of R is called a pair of *pants* of R if P is triply connected and if every connected component of the relative boundary of P in R is a simple closed geodesic on R.

Fix a pair of pants P of R. Let Γ be a Fuchsian model of R acting on Δ, $\pi : \Delta \rightarrow R = \Delta/\Gamma$ be the projection. Let $\tilde{P}$ be a connected component of $\pi^{-1}(P)$. The subgroup $\Gamma_{\tilde{P}} := \{ \gamma \in \Gamma : \gamma(\tilde{P}) = \tilde{P} \}$ is a free group generated by 2 hyperbolic transformations, and $P = \tilde{P}/\Gamma_{\tilde{P}}$.

Let $\hat{P} = \Delta/\Gamma_{\tilde{P}}$. Then $\hat{P}$ is a triply connected surface obtained from P by attaching a suitable doubly connected region along each boundary component. $\Gamma_{\tilde{P}}$ is a Fuchsian model of $\hat{P}$. Therefore, P is the unique pair of pants of $\hat{P}$ and is called the *Nielsen kernel* of $\hat{P}$. $\hat{P}$ is called the *Nielsen extension* of P.

Existence and Uniqueness of Pants

The relationship between the complex structure of a triply connected domain Ω and the hyperbolic structure of the unique pair of pants P of Ω (induced by the hyperbolic metric on Ω) will be considered.

Let L_1, L_2 and L_3 be the boundary components, which are simple closed geodesics, of the pair of pants P. Let Γ_0 be a Fuchsian model of Ω acting on Δ. Then Γ_0 is a free group generated by 2 hyperbolic transformations γ_1 and γ_2. It is assumed that γ_1 and γ_2 cover L_1 and L_2 respectively.

Theorem 2.4. *For any given triple (a_1, a_2, a_3) of positive number, there exists a triply connected planar Riemann surface Ω such that $l(L_j) = a_j$ where $j = 1,2,3$.*

Proof. Let C_1 be the part of the imaginary axis in Δ. Fix another geodesic C_2 on Δ such that the Poincaré distance between C_1 and C_2 is equal to $a_1/2$. Geodesics on Δ from which the Poincaré distance to C_1 are equal to $a_3/2$ form a real one-parameter family. Hence there exists a geodesic C_3 in this family such that the Poincaré distance between C_2 and C_3 is equal to $a_2/2$.

Let $\{ z_1,z_2 \}$, $\{ z_3,z_4 \}$ and $\{ z_5,z_6 \}$ be the pairs of the points in D uniquely determined by the conditions

$$\rho(z_1,z_2) = a_1/2 \qquad z_1 \in C_1, z_1 \in C_2,$$

$$\rho(z_3,z_4) = a_2/2 \qquad z_3 \in C_2, z_4 \in C_3,$$

$$\rho(z_5,z_6) = a_3/2 \qquad z_5 \in C_3, z_6 \in C_1.$$

respectively. Let L'_1, L'_2 and L'_3 be the geodesics connecting the pairs $\{ z_1,z_2 \}$, $\{ z_3,z_4 \}$ and $\{ z_5,z_6 \}$ respectively.

Let D be the hyperbolic hexagon bounded by $\{ C_j, L'_j \}_{j=1}^{3}$. Let η_j be the reflection with respect to C_j j=1,2,3. $\gamma_1 := \eta_1 \circ \eta_2$ and $\gamma_2 := \eta_3 \circ \eta_1$ are the hyperbolic elements of Aut(Δ). Let Γ_0 be the group generated by γ_2 and γ_2.

Then $\Omega = \Delta/\Gamma_0$ is triply connected, and that the unique pair of pants P of Ω is the interior of the set obtained by identifying the boundary of $D \cup \eta_1(D)$ under the action by Γ_0.

$\square$

Theorem 2.5. *The complex structure of a pair of pants P is uniquely determined by the hyperbolic lengths of the ordered boundary components of P.*

Proof. see [Ke]. Suppose that a pair of pants of P is given. Let a_j be the hyperbolic length of the boundary component L_j (j=1,2,3) of P, and Γ_0 be a Fuchsian model of $\hat{P}$ acting on H. Let $\{ \gamma_1,\gamma_2 \}$ be a system of generators of Γ_0. Assume γ_k cover L_k (k=1,2,3) where $\gamma_3 := (\gamma_2 \circ \gamma_1)^{-1}$.

It suffices to show that γ_1 and γ_2 are uniquely determined by $\{ a_j \}_{j=1}^{3}$.

Assume

$$\gamma_1(z) = \lambda^2 z \qquad 0 < \lambda < 1,$$

$$\gamma_2(z) = \frac{az+b}{cz+d} \qquad ad-bc = 1,\ c > 0, \text{ and}$$

that 1 is the attractive fixed points of γ_2.

Then $\gamma_2(\infty) = a/c > 0$ and $a+d > 0$, since the middle-point $\frac{a-d}{2c}$ of 2 fixed points of γ_2 has a value less than $\gamma_2(\infty)$.

Write $\gamma_3^{-1}(z) = \frac{pz+q}{rz+s}$ $\quad ps - qr = 1$. Assume $[\, p, q, r, s \,] = [a\lambda, b/\lambda, c\lambda, d/\lambda\,]$. The middle-point $\frac{p-s}{2r}$ of 2 fixed points of γ_3^{-1} has a value greater than $\gamma_3^{-1}(\infty)$. So $p + s < 0$.

By proposition 2.3, we have

$$(\lambda + \frac{1}{\lambda})^2 = 4 \cosh^2(a_1/2),$$

$$(a+d)^2 = 4 \cosh^2(a_2/2),$$

$$(p+s)^2 = 4 \cosh^2(a_3/2).$$

Therefore, γ_1 and γ_2 are uniquely determined by $\{ a_j \}_{j=1}^{3}$.

□

Corollary. *Every pair of pants of P has an anti-holomorphic automorphism J_P of order 2. Moreover, $F_{J_P} = \{ z \in P : J_P(z) = z \}$ consists of 3 geodesics $\{ D_j \}_{j=1}^{3}$ in P satisfying :*

For every j=1,2,3, D_j has the endpoints on, and is orthogonal to, both L_j and L_{j+1}, where $L_4 = L_1$.

2. Fenchel-Nielsen Coordinates

Pant Decomposition

Fix $[R,\Sigma]$ in T_g. A set L of mutually disjoint simple closed geodesics on a Riemann surface R of genus $g \geq 2$ is called *maximal* if there is no set L' which includes L properly. A maximal set $L := \{ L_j \}_{j=1}^N$ of mutually disjoint simple closed geodesics on R is called *a system of decomposing curves*, and the family $P = \{ P_k \}_{k=1}^M$ consisting of all connected components of $R \setminus \cup_{j=1}^N L_j$ the *pants decomposition* of R corresponding to L. It can be checked that $N = 3g-3$ and $M = 2g-2$.

Geodesic Length Functions

Fix $[R,\Sigma]$ in T_g, and a system $L = \{ L_j \}_{j=1}^N$ of decomposing curves on R. For every t in T_g, the point in T_g corresponding to t is denoted by $[R_t, \Sigma_t]$. Then, a system $L_t = \{ L_j(t) \}_{j=1}^N$ of decomposing curves on R_t can be uniquely determined, where $L_j(t)$ is the projection of the axis of an element of the Fuchsian model of R_t which covers $f_t(L_j)$ for every j.

For every $t \in F_g$ and j, $l_j(t) := l(L_j(t))$ is called the *geodesic length function* for L_j.

By proposition 2.3, every length function $l_j(t)$ is real analytic on F_g.

Twisting Parameters

For all j, let $P_{j,1}$ and $P_{j,2}$ be 2 pairs of pants in P having L_j as a boundary components. Let J_1 and J_2 be the reflection of $P_{j,1}$ and $P_{j,2}$ respectively. Take a fixed point $c_{j,k}$ of J_k on L_j for each $P_{j,k}$ (k=1,2). An orientation on L_j is fixed. Each $c_{j,k}$ is the end point on L_j of the geodesic $D_{j,k}$ joining L_j and another boundary component $L_{j,k}$ in $P_{j,k}$.

For all t and j, let $P_{j,1}(t)$ and $P_{j,2}(t)$ be the connected components of $R_t \setminus \cup_{j=1}^N L_j(t)$ corresponding to $P_{j,1}$ and $P_{j,2}$, respectively. Let $L_{j,k}(t)$ be the boundary component of $P_{j,k}(t)$ corresponding to $L_{j,k}$ and $D_{j,k}(t)$ be the geodesic joining $L_j(t)$ and $L_{j,k}(t)$ in $P_{j,k}(t)$, and by $c_{j,k}(t)$ the point of $D_{j,k}(t)$ on $L_j(t)$. Then each $c_{j,k}(t)$ is a fixed point of the reflection of $P_{j,k}(t)$.

Let $T_j(t)$ be the oriented arc on $L_j(t)$ from $c_{j,1}(t)$ to $c_{j,2}(t)$. $\theta_j(t) := 2\pi \frac{l(T_j(t))}{l_j(t)}$, which is well-defined modulus 2π, is called the *twisting parameter* with respect to L_j.

Fenchel-Nielsen Coordinates

A real-analytic mapping $\psi : F_g \to (R^+)^{3g-3} \times (S^1)^{3g-3}$ is defined by

$$\psi(t) = (l_1(t),\ldots l_{3g-3}(t), \exp(i\theta_1(t)),\ldots \exp(i\theta_{3g-3}(t))).$$

Lemma 2.6. *Fix a single-valued continuous branch of the twisting parameter $\theta_j(t)$ on F_g for all j. Then $\tilde{\psi}(t) = (l_1(t),\ldots l_{3g-3}(t),\theta_1(t),\ldots\theta_{3g-3}(t))$ is real-analytic on F_g which is called Fenchel-Nielsen coordinates of T_g associated with the system L of decomposing curves.*

Theorem 2.7. *$\tilde{\psi} : F_g \to (R^+)^{3g-3} \times R^{3g-3}$ is a homeomorphism. In particular, $\tilde{\psi}$ gives a system of global coordinates on F_g, and hence on T_g.*

Proof. A detailed proof is given for example in [Im,Ta].

Remark. $\tilde{\psi} : F_g \to (\mathbf{R}^+)^{3g-3} \times \mathbf{R}^{3g-3}$ is a diffeomorphism.

3. Fricke-Klein Embedding

We shall describe a set of simple closed geodesics on a Riemann surface of genus $g \geq 2$ whose hyperbolic lengths determine the surface.

Fix a Riemann surface R of genus g and a system L of decomposing curves on R. Let P be the pants decomposition of R corresponding to L. For all $L_j \in L$, let $P_{j,1}$ and $P_{j,2}$ be the elements of P having L_j as a boundary component.

For all j, $W_j := P_{j,1} \cup L_j \cup P_{j,2}$. A simple closed geodesic Δ_j^o , which intersect L_j, is fixed in W_j. Let Δ_j' be the unique simple closed geodesic which is freely homotopic to the simple curve obtained from Δ_j^o by applying the Dehn twist with respect to L_j.

For all $t \in F_g$, let $[\, R_t,\Sigma_t \,]$ be the corresponding point of T_g. For any closed geodesic L on R, L(t) is expressed as the corresponding geodesic on R_t.Set

$$l_j(t) := l(L_j(t)) \quad l_{3g-3+j}(t) := l(\Delta_j^o(t)) \qquad l_{6g-6+j}(t) := l(\Delta_j'(t))$$

$\forall\, j = 1,\dots 3g\text{-}3$, and

$$\tilde{l}(t) := (l_1(t),\dots l_{9g-9}(t)).$$

Lemma 2.8. *Let L_1 and L_2 be mutually disjoint geodesics in H. Then $\rho(z,w)$ is strictly convex on $\{\, (z,w) \;: z \in L_1,\, w \in L_2 \,\}$.*

Fix t_o in F_g, and $\tilde{\psi}(t_o) := (a_1,\dots a_{3g-3},\alpha_1,\dots \alpha_{3g-3}) \in (\mathbf{R}^+)^{3g-3} \times \mathbf{R}^{3g-3}$. Fix j and for all $s \in \mathbf{R}$, $t(s) := \psi^{-1}(a_1,\dots a_{3g-3},\alpha_1,\dots \alpha_{j-1},\alpha_j+s,\dots \alpha_{3g-3})$.

Proposition 2.9. *The positive function $f(s) := l(\Delta_j^o(t(s)))$ is strictly convex and proper on **R**. In particular, there exists a unique value of s at which f attains its minimum.*

Proof. Let Γ_o be the Fuchsian group represented by t_o, and such that $\gamma_j \in \Gamma_o$ covers $L_j(t_o)$. On the axis of A_j of γ_j , fix a point z_o which is projected to an intersection point p of $L_j(t_o)$ and $\Delta_j^o(t_o)$. Suppose that $\delta_j^o \in \Gamma_o$ covers $\Delta_j^o(t_o)$ and whose axis B_j passes through z_o. By the assumption, the projection of the geodesic I contained in B_j and connecting z_o to $z_o' := \delta_j^o(z_o)$ intersect $L_j(t_o)$ at q. Let z_o be the lift of q on I, and A_j' be the lift of $L_j(t_o)$ passes through z_1.

For any z on an oriented geodesic L on H and any $\alpha \in R$, let $z(\alpha)_L$ be the point on L obtained by translating z in the positive direction along L by hyperbolic length α. Since $t(s)$ represents the marked surface obtained from the surface represented by t_o by twisting along $L_j(t_0)$ by hyperbolic length $\frac{1}{2\pi}$ $l(L_j(t_0))\, s = \hat{s}$, we have

$$f(s) = \inf_{z \in A_j \; w \in A'_j} \{\rho(z,w) + \rho(w\hat{s}, \delta_j^o(z)\hat{s})\}.$$

A strictly convex function $F : A_j \times A'_j \times \mathbf{R} \to \mathbf{R}^+$ is defined by

$$F(z,w,\hat{s}) = \rho(z,w(-\hat{s})) + \rho(w,\delta_j^o(\hat{s})).$$

Since $\rho(z,w) \to +\infty$ as either z or w tend to ∂H, F is proper.

□

Theorem 2.10. *$\mathcal{T} : F_g \to (R^+)^{9g-9}$ is a proper embedding.*

Proof. Since $\Delta'_j(t)$ is freely homotopic to the curve obtained from $\Delta_j^o(t)$ by applying the Dehn twist along $L_j(t)$ for all j, $l(\Delta'_j(t(s))) = l(\Delta_j^o(t(s+2\pi))) = f(s+2\pi)$.

Since $f(s)$ is strictly convex and proper on R, the mapping $g : \mathbf{R} \to \mathbf{R}^2$ by

$$g(s) = (f(s), f(s+2\pi))$$

is proper, injective, continuous. Hence $g(s)$ is a proper embedding of $\mathbf{R}$ into $(\mathbf{R}^+)^2$.

Since j is arbitrary, the assertion follows from theorem 2.7.

□

CHAPTER 3.

Quasiconformal Mappings

1. Definitions

Analytic Definitions of Quasiconformal Mappings

Definition A. Let $f : D \rightarrow \mathbf{C}$ be an orientation-preserving homeomorphism of a domain D in **C**. f is said to be *quasiconformal* (qc) on D if

1. f is *absolutely continuous on lines* (ACL) on D. This means for any rectangle $[a,b] \times [c,d]$ in D, $f(x,y)$ is absolutely continuous on $[a,b]$ as a function of x for almost every $y \in [c,d]$, and $f(x,y)$ is absolutely continuous on $[c,d]$ as a function of y for almost every $x \in [a,b]$.

2. There exists a constant $k \in [0,1)$ such that $| f_{\bar{z}} | \leq k | f_z |$ almost everywhere on D. Observe that by (1), f_z and $f_{\bar{z}}$ are well-defined and finite at almost every $z \in D$.

f is said to be *K-qc* on D where $K := \frac{1+k}{1-k}$. We call the infimum of K such that f is K-qc the *maximal dilatation* of f K_f. We quote the following propositions which classify the meaning of f_z and $f_{\bar{z}}$. The first proposition is due to Gehring and Lehto. [Im,Ta]

Proposition 3.1. *If a homeomorphism $f : D \rightarrow \mathbf{C}$ has the partial derivatives f_x and f_y almost everywhere on D, then f is totally differentiable almost everywhere on D.*

Proposition 3.2. *If f is quasiconformal on D, then f_z and $f_{\bar{z}}$ are locally square integrable on D.*

For all quasiconformal mapping f of a domain D, f_z and $f_{\bar{z}}$ are coincident with those in the sense of distribution.

Definition A'. Let $f : D \to \mathbf{C}$ be an orientation-preserving homeomorphism of a domain D in **C**. f is said to be *quasiconformal* (qc) on D if

1'. The distributional partial derivatives of f with respect to z and $\bar{z}$ can be represented by locally integrable functions f_z and $f_{\bar{z}}$, on D respectively.

2. There exists a constant $k \in [0,1)$ such that $| f_{\bar{z}} | \le k | f_z |$ almost everywhere on D.

Theorem 3.3. *Definition A and A' of quasiconforrmal mappingss are equivalent.*

Proof. $\Rightarrow$ The result follows from the proposition 3.2 .

$\Leftarrow$ It suffices to show that f is ACL.

Fix a rectangle $R = [a,b] \times [c,d]$ in D. Fix a point $R(x_0)$ on [a ,b] and $x_0 := [a,x_0] \times [c,d]$. The distributional partial derivatives of f with respect to x. Take $\varphi_1(x)\,\varphi_2(y) \in C_0^\infty(R(x_0))$. Then

$$\iint_{R(x_0)} [f_x](x,y)\,\varphi_1(x)\,\varphi_2(y)\,dxdy = -\iint_{R(x_0)} f(x,y)\,\varphi'_1(x)\,\varphi_2(y)\,dxdy.$$

Let $\varphi_2(y)$ tend monotonously to the characteristic function of (c,η), and then

$$\int_c^\eta\int_a^{x_0} [f_x](x,y)\,\varphi_1(x)\,dxdy = -\int_c^\eta\int_a^{x_0} f(x,y)\,\varphi'_1(x)\,dxdy.$$

Since η is arbitrary, $\int_a^{x_0}[f_x](x,y)\,\varphi_1(x)\,dx = -\int_a^{x_0} f(x,y)\,\varphi'_1(x)\,dx$ for almost every y on [c,d].

For every sufficiently large n, choose φ_1 to be equal to 1 on $[\,a+\frac{1}{n}, x_0-\frac{1}{n}\,]$, increasing on $[\,a, a+\frac{1}{n}\,]$ and decreasing on $[\,x_0-\frac{1}{n}, x_0\,]$. Letting $n \to \infty$, we have

$$\int_a^{x_0}[f_x](x,y)\,dx = f(x_0,y) - f(a,y) \qquad \text{almost everywhere } y \in [c,d]. \quad (*)$$

The exceptional set of y depends on x_0. Since $E := [a,b] \cap \mathbf{Q}$ is countable, (*) holds almost everywhere on [c,d] for all $x_0 \in E$. Since both sides of (*) are continuous with respect to x_0, and E is dense on [a,b], (*) holds almost everywhere on [c,d] for all $x_0 \in [a,b]$. For all y such that (*) holds for every x_0, f(x,y) is absolutely continuous with respect to x, and $[f_x]$ coincides with f_x almost everywhere on [a,b].

The assertion is similar on the partial derivative of f with respect to y. Thus f is ACL on D.

□

Applying Weyl's lemma to the theorem, we get

Corollary. *A 1-qc mapping is conformal.*

Geometrical Definition of Quasiconformal Mappings

A *quadrilateral* is a pair $(Q;q_1,q_2,q_3,q_4)$ of a Jordan closed domain Q and 4 points q_1,q_2,q_3 and q_4 on ∂Q which are mutually distinct and located in this order with respect to positive orientation of ∂Q. By Riemann's mapping theorem and the Schwarz-Christoffel transformation, we get

Proposition 3.4. *For any quadrilateral $(Q;q_1,q_2,q_3,q_4)$, there exists a homeomorphism $h : Q \to [0,a] \times [0,b]$ which is conformal in intQ and satisfies*

$$h(q_1) = 0, \qquad h(q_2) = a, \qquad h(q_3) = a + ib, \qquad \text{and} \qquad h(q_4) = ib.$$

Moreover, the modulus of the quadrilateral $M(Q) := \frac{a}{b}$ is independent of h.

Concerning the moduli of quadrilaterals, we have the following estimates.

Lemma 3.5. *Every K-qc mapping f of a domain D satisfies*

$$\frac{M(Q)}{K} \leq M(f(Q)) \leq K\,M(Q) \qquad \forall \text{ quadrilateral } Q.$$

Using the moduli of quadrilaterals, we can define quasiconformal mappings without using partial derivatives.

Definition G. Let f be an orientation-preserving homeomorphism of a domain D into **C**. f is quasiconformal on D if f satisfies

3. There exists a constant $K \geq 1$ such that $M(f(Q)) \leq K\, M(Q)$ $\forall$ quadrilateral Q in D.

Theorem 3.6. *Definition A and G of quasiconformal mappingss are equivalent.*

Proof. $\Rightarrow$ By lemma 3.5.

$\Leftarrow$ To show that f is ACL on D.

Fix $R := [a,b] \times [c,d]$ in D. $F(y) :=$ area of $f([a,b] \times [c,y])$ $\forall$ $y \in [c,d]$. Since F(y) is a non-decreasing function, F is differentiable at almost every $y \in [c,d]$. Fix such $y = y_0$, and $\eta := y - y_0 > 0$.

Take a family $\{ I_j \}_{j=1}^{m}$ of mutually disjoint intervals of [a,b], and $R_j := I_j \times [y_0,y]$ and $Q_j := f(R_j)$ $\forall$ j. Let l_j and l'_j be the lengths of I_j and $f(I_j \times \{y_0\})$ respectively.

Fix j and $\varepsilon > 0$. Then $\exists$ a set $\{ \zeta_k \}_{k=1}^{n}$ of points on $f(I_j \times \{y_0\})$ such that

$$\zeta_1 = f(a_j + iy_0),\ \zeta_n = f(b_j + iy_o) \text{ and } \sum_{k=1}^{n} |\zeta_k - \zeta_{k-1}| \geq l'_j - \frac{\varepsilon}{2}$$

where $I_j := [a_j,b_j]$ and $\sum_{k=1}^{n} |\zeta_k - \zeta_{k-1}| \geq \frac{1}{\varepsilon}$ when $l'_j = \infty$.

Since f is uniformly continuous on R, take η sufficiently small, we assume that

$$| f(x_0 + i(y_0+\xi)) - f(x_0 + iy_0) | \leq \frac{\varepsilon}{4n} \quad \forall\, x_0 \in [a,b],\ \forall\, \xi \in [o,\eta].$$

Take any curve L in R_j connecting 2 sides of ∂R_j which are parallel to the x-axis.

Then the length of f(L) is not less than $\tilde{l}_j := \sum_{k=1}^{n} |\zeta_k - \zeta_{k-1}| - \frac{\varepsilon}{2}$.

On the other hand, let $\tilde{h} : \tilde{R}_j := [\tilde{a}_j,\tilde{b}_j] \times [\tilde{c}_j,\tilde{d}_j] \rightarrow Q_j$ be a homeomorphism and be conformal in int$\tilde{R}_j$. Then

$$\tilde{l}_j^2 \leq (\tilde{b}_j - \tilde{a}_j) \int_{\tilde{a}_j}^{\tilde{b}_j} |\tilde{h}_j|^2 \, dx.$$

Integrating both sides with respect to y on $[\tilde{c}_j,\tilde{d}_j]$, we have $\tilde{l}_j 2 \leq M(Q_j) \times$ area of Q_j.

Since $M(Q_j) \leq K\, M(R_j) = K\frac{l_j}{\eta}$, we have

$$\left(\sum_{j=1}^{n} \tilde{l}_j\right)^2 \leq K \frac{F(y) - F(y_0)}{y - y_0} \sum_{j=1}^{n} l_j .$$

Since F is differentiable at y_0, $\frac{F(y) - F(y_0)}{y - y_0}$ converges as $\eta \rightarrow 0$. In particular, every l'_j is finite. Since $\tilde{l}_j \geq l'_j - \varepsilon$, we have $\sum_{j=1}^{m} l'_j \rightarrow 0$ as $\sum_{j=1}^{m} l_j \rightarrow 0$. Thus $f(x,y_0)$ is absolutely continuous on [a,b].

Similarly, $f(x_0,y)$ is absolutely continuous on [c,d] for almost every $x_0 \in$ [a,b]. Thus f is ACL on D.

2. We have to show that $| f_{\bar{z}} | \leq k | f_z |$ almost everywhere on D where $k = \frac{K-1}{K+1}$.

By (1), f is totally differentiable at almost every z in D. Fix $z = z_0$. Assume $f_{\bar{z}}(z_0) \geq 0$. Then $f_z \geq 0$. If $f_z = 0$, the assertion is true.

If $f_z = 0$, then consider $f(z) = f(z_0) + f_z(z_0)\, z + f_{\bar{z}}(z_0)\, \bar{z} + o(|z|)$ in the neighborhood of z_0. Let $R_\varepsilon = [0,\varepsilon] \times [0,\varepsilon]\ \forall\ \varepsilon > 0$.

Then $f(R_\varepsilon)$ is approximately the rectangle $[\ a,\ a + (f_z(z_0)\, z + f_{\overline{z}}\,(z_0)\, \overline{z}\,)\ \varepsilon\ ]$ $\times\ [\ b,\ b + (f_z(z_0)\, z - f_{\overline{z}}\,(z_0)\, \overline{z}\,)\ \varepsilon\ ]$ where $f(z_0) = a + ib$.

By Rengel's inequality [Le], $K \geq M(f(R_\varepsilon)) \geq \dfrac{f_z(z_0)\, z + f_{\overline{z}}\,(z_0)\, \overline{z}}{f_z(z_0)\, z - f_{\overline{z}}\,(z_0)\, \overline{z}} + o(1).$

Therefore, $\quad |f_{\overline{z}}\,| \leq k\,|\,f_z\,| \quad$ when $\varepsilon \to 0$.

$\square$

Using the equivalence of Definitions A, A' & G, we can prove the following results.

Theorem 3.7.

1. The inverse of a K-qc mapping is K-qc.

2. K-quasiconformality is conformal invariant.

3. $\forall K_1$-qc mapping f of a domain D and $\forall K_2$-qc mapping of $f(D)$, $g \circ f$ is K_1K_2-qc.

Proposition 3.8. *If f is quasiconformal on a domain D, then $f_z \neq o$ almost everywhere on D.*

Thus, for every quasiconformal mapping f of a domain D, $\mu_f = \dfrac{f_{\overline{z}}}{f_z}$ is defined almost everywhere on D. μ_f is called t he *complex dilatation* of f on D.

Proposition 3.9. *For all quasiconformal mapping f and g of a domain D, $\mu_{g o f^{1}} \circ f = \dfrac{f_z}{\overline{f_z}}\,\dfrac{\mu_g - \mu_f}{1 - \overline{\mu_f}\,\mu_g}$ almost everywhere on D.*

2. Existence Theorem on Quasiconformal Mappings

We shall show that for every measurable μ with $\|\mu\|_\infty < 1$, there exists a quasiconformal mapping whose complex dilatation is equal to μ.

Let $L^\infty(D)$ be the complex Banach space of all bounded measurable functions on a domain D in **C**. The norm is given by

$$\|\mu\|_\infty = \operatorname*{ess.\,sup}_{z \in D} |\mu(z)|, \quad \mu \in L^\infty(D).$$

Let $B(D)_1 = \{ \mu \in L^\infty(D) : \|\mu\|_\infty < 1 \}$. Elements of $B(D)_1$ will be called *Beltrami coefficient* on D.

We note that a quasiconformal mapping with the prescribed complex dilatation is unique as follows.

Proposition 3.10. *Let $\mu \in B(D)_1$. Suppose f and g are qc mappings on D. Then $\mu_f = \mu_g = \mu \Leftrightarrow g \circ f^{-1}$ is a conformal mapping on $f(D)$.*

For any $\mu \in B(\mathbf{C})_1$, we consider how to solve the Beltrami differential equation

$$f_{\bar z} = \mu f_z.$$

The following classical *Pompeiu's formula* is to solve the $\bar\partial$ problem.

Proposition 3.11. *Fix p with $2 < p < \infty$. Let f be a continuous function on* **C** *whose distributional partial derivatives are represented by $f_{\bar z}$ and f_z of $L^p(\mathbf{C})$. Then*

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(z)dz}{z - \zeta} - \frac{1}{\pi}\iint_B \frac{f_{\bar z}(z)}{z - \zeta}\, dxdy, \qquad \zeta \in B$$

for every open disk B in **C**.

We define a linear operator P on $L^p(\mathbf{C})$ by setting

$$Ph(\zeta) = -\frac{1}{\pi}\iint_{\mathbf{C}} h(z)\left(\frac{1}{z-\zeta} - \frac{1}{z}\right) dxdy, \qquad h \in L^p(\mathbf{C}), \qquad \zeta \in \mathbf{C}.$$

Lemma 3.12. *For every p with $2 < p < \infty$ and for every $h \in L^p(\mathbf{C})$, Ph is a uniformly Hölder continuous function on* **C**, *with exponent $(1 - \frac{2}{p})$, and satisfies*

$Ph(0) = 0$.

Moreover, Ph satisfies

$$(Ph)_{\bar{\zeta}} = h$$

*on **C** in the sense of distribution.*

Next, we need to obtain a suitable representation for $(Ph)_z$. For $h \in C_0^\infty(\mathbf{C})$, we have the Green's formula

$$(Ph)_\zeta(\zeta) = \lim_{\varepsilon \to 0} \frac{-1}{2\pi i}\left[\int_{|z-\zeta| = \varepsilon} \frac{h(z)}{z-\zeta} d\bar{z} - \iint_{|z-\zeta| > \varepsilon} \frac{h(z)}{(z-\zeta)^2} dz \wedge d\bar{z}\right]$$

The first term converges to 0 as $\varepsilon \to 0$. Let T be the linear operator T defined by

$$Th(\zeta) = \lim_{\varepsilon \to 0}\left[-\frac{1}{\pi}\iint_{|z-\zeta| > \varepsilon} \frac{h(z)}{(z-\zeta)^2} dx \wedge dy\right] \qquad h \in C_0^\infty(\mathbf{C}).$$

Lemma 3.13. *Every $h \in C_0^\infty(\mathbf{C})$ satisfies*

$$(Ph)_z = Th \quad \text{on } \mathbf{C}, \quad \text{and} \quad \|Th\|_2 = \|h\|_2.$$

The last equality implies that T is extended to a bounded linear operator on $L^2(\mathbf{C})$ into $L^2(\mathbf{C})$ with norm 1. We have considered P as that on $L^p(\mathbf{C})$ with p > 2; it turns out by the classical *Calderòn-Zygmund's theorem* that T also extends to a bounded linear operator on $L^p(\mathbf{C})$ into $L^p(\mathbf{C})$ with p > 2 as follows.

Theorem 3.14.(Calderòn-Zygmund) *For every p with $2 \le p < \infty$,*

$$C_p = \sup_{h \in C_0^\infty(C)\, \|h\|_p = 1} \|Th\|_p$$

is finite. Hence, the operator T is extended to a bounded linear operator of $L^p(\mathbf{C})$ into $L^p(\mathbf{C})$ with norm C_p.

Moreover, C_p is continuous with respect to p. In particular, C_p satisfies

$$\lim_{p \to 2} C_p = 1.$$

We shall assume this theorem (a detailed proof can be found in [Im,Ta]).

In particular, we get

Proposition 3.15. *For any $p > 2$ and every $h \in L^p(\mathbf{C})$,*

$$(Ph)_z = Th$$

*on **C** in the sense of distribution.*

Existence of the Normal Solution

Theorem 3.16. *Fix $k \in [0, 1)$. Take $p > 2$ with $kC_p < 1$. Then for any $\mu \in B(\mathbf{C})_1$ with $\|\mu\|_\infty \leq k$ and with compact support, there exists a unique continuous function f such that $f(0) = 0$, $(f_z - 1) \in L^p(\mathbf{C})$, and*

$$f_{\bar{z}} = \mu f_z$$

*on **C** in the sense of distribution.*

Sketch of proof. Lemma 3.12 says that the operator solves the $\bar{\partial}$-problem. By the normalization conditions in the theorem, we get the formula

$$f = P(f_{\bar{z}}) + z = P(\mu f_z) + z \tag{1}$$

Then by proposition 3.15, we have

$$f_z = T(\mu f_z) + 1 \tag{2}$$

Iterating (2) gives us

$$f_z = 1 + T\mu + T(\mu T\mu) + T(\mu T(\mu T\mu)) + \ldots$$

Substituting back in (1) gives

$$f = P(\mu(1 + T\mu + T(\mu T\mu) + \ldots) + z.$$

□

The function f is called the *normal solution* of the Beltrami equation for μ. Here we state some properties of the normal solution.

Corollary 1.

$$\|f_{\bar{z}}\|_p \leq \frac{\|\mu\|_p}{1 - kC_p},$$

and

$$|f(\zeta_1) - f(\zeta_2)| \leq \frac{K_p}{1 - kC_p} \|\mu\|_p \, |\zeta_1 - \zeta_2|^{1-2/p} + |\zeta_1 - \zeta_2|$$

for every $\zeta_1, \zeta_2 \in \mathbf{C}$ and K_p is a constant.

Corollary 2. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in $B(\mathbf{C})_1$ satisfying the following conditions:*

1. $\|\mu_n\|_\infty < k$ *for every n,*

2. *every μ_n has a support in $\{ z \in \mathbf{C} : |z| < M \}$ for some constant M, and*

3. *μ_n converges to some $\mu \in B(\mathbf{C})_1$ almost everywhere on **C** as $n \to \infty$.*

*Let f_n and f be the normal solutions for μ_n and μ respectively. Then $f_n \to f$ uniformly on **C** as $n \to \infty$, and*

$$\lim_{n \to \infty} \|(f_n)_z - f_z\|_p = 0.$$

Choose a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $C_0^\infty(\mathbf{C})$ with $\|\mu_n\|_\infty \leq k$ such that the support of μ_n is contained in $\{ z \in \mathbf{C} : |z| < M \}$ for every n, and that $\mu_n \to \mu$ almost everywhere on **C** as $n \to \infty$. Let f_n be the normal solution for μ_n for every n.

By a generalization of Weyl's lemma, it can be shown that f_n is C^1. Applying inverse mapping theorem, we get that f_n is locally homeomorphism on **C**. Since $f = P(\mu(1 + T\mu \ldots)) + z$, f_n has a simple pole at $z = \infty$. Thus f_n is locally homeomorphism on $\hat{\mathbf{C}}$.

Lemma 3.17. *If a function $f_n : \hat{C} \to \hat{C}$ is locally homeomorphic, then f_n is homeomorphic.*

By the lemma, we get f_n homeomorphic on $\hat{\mathbf{C}}$

Lemma 3.18. *Every f_n satisfies*

$$|\zeta_1 - \zeta_2| \le \frac{K_p}{(1 - kC_p)^{1+2/p}} ||\mu_n||_p \, |f_n(\zeta_1) - f_n(\zeta_2)|^{1-2/p} + |f_n(\zeta_1) - f_n(\zeta_2)|$$

for every $\zeta_1, \zeta_2 \in \mathbf{C}$ and K_p is a constant.

Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ be as defined before. Then f_n converges to f uniformly on **C**. Since $||\mu_n||_p \to ||\mu||_p$, we can replace f_n and μ_n by f and μ, respectively. Hence f is a homeomorphism on **C**. Since $f_z - 1 \in L^p(\mathbf{C})$, so does $f_{\bar{z}} = \mu f_z$. Thus f satisfies the assumptions in Definition A', and hence f is quasiconformal.

Existence Theorem

Theorem 3.19. *For every $\mu \in B(\mathbf{C})_1$, there exists a homeomorphism $f : \hat{C} \to \hat{C}$ which is a qc mapping of **C** with complex dilatation μ.*

Moreover, f is uniquely determined by the normalization conditions:

$$f(0) = 0, \qquad f(1) = 1, \text{ and } f(\infty) = \infty.$$

The function f is called the *canonical μ-qc mapping of* $\hat{\mathbf{C}}$ and is denoted by f^{μ}.

Proof. The uniqueness follows by proposition 3.10 and the normalization conditions. We have to show the existence of f^{μ}.

Case(1): Suppose μ has a compact support. Let F be the normal solution for μ. Then

$$f^{\mu}(z) = \frac{F(z)}{F(1)} \text{ is the desired solution.}$$

Case(2). For $\mu = 0$ almost everywhere in some neighborhood of the origin. Set

$$\tilde{\mu}(z) = \mu(\frac{1}{z}) \left(\frac{\overline{z}}{z}\right)^2 \qquad \text{almost everywhere on } \mathbf{C}.$$

Then $\tilde{\mu} \in B(\mathbf{C})_1$ has a compact support. So there exists the canonical $\tilde{\mu}$-qc mapping $f^{\tilde{\mu}}$ of $\hat{\mathbf{C}}$. By proposition 3.1, $f^{\tilde{\mu}}$ is totally differentiable almost everywhere on $\mathbf{C}$. At every such point of $1/z$, the quasiconformal mapping

$$f(z) = \frac{1}{f^{\tilde{\mu}}(1/z)}$$

is totally differentiable so that we a can apply the chain rule. Thus we have

$$\mu_f = \tilde{\mu}(\frac{1}{z}) \left(\frac{\overline{z}}{z}\right)^2 = \mu(z) \qquad \text{almost everywhere on } \mathbf{C}.$$

So f satisfies the normalization conditions and is the desired solution.

Finally, suppose that μ is a general Beltrami coefficient. In this case, we set

$$\mu_1 = \begin{cases} \mu(z) & z \in \mathbf{C}\setminus\overline{\Delta} \\ 0 & z \in \overline{\Delta}, \end{cases}$$

Then we have shown that f^{μ_1} exists. Moreover, we set

$$\mu_2 = \left(\frac{f^{\mu_1}{}_z}{\overline{f^{\mu_1}{}_z}} \frac{\mu - \mu_1}{1 - \overline{\mu_1}\,\mu} \right) \circ (f^{\mu_1})^{-1}.$$

Since μ_2 has a compact support, f^{μ_2} exists. $g := f^{\mu_2} \circ f^{\mu_1}$ is qc, and $\mu_g = \mu$ almost everywhere. So g satisfies the normalization conditions and is the desired solution.

□

Then we state several applications of the existence theorem.

Proposition 3.20. *Every quasiconformal mapping of Δ onto a Jordan domain D is extended to a homeomorphism of $\overline{\Delta}$ onto $\overline{D}$.*

Proposition 3.21. *There exists no quasiconformal mappings of Δ onto* **C**.

Proposition 3.22. *Let $\mu \in B(H)_1$. Then there exists a quasiconformal mapping $w : H \to H$ with complex dilatation μ.*

Moreover, w is uniquely determined by the normalization conditions:

$$w(0) = 0, \qquad w(1) = 1, \qquad \text{and} \qquad w(\infty) = \infty .$$

The function w is called the *canonical μ-qc mapping of* H and is denoted by w^{μ}.

3. Dependence on Beltrami Coefficients

Some of the most important and useful facts on the canonical quasiconformal mapping f^{μ} of $\hat{\mathbf{C}}$ concern dependence of f^{μ} on the Beltrami coefficient μ.

Lemma 3.23. *For every $p > 2$,*

$$\|(f^{\mu})_z - 1\|_{p,\Delta} \to 0 \text{ as } \|\mu\|_{\infty} \to 0,$$

where $\mu \in B(C)_1$ and $\|h\|_{p,\Delta} = \left[\iint_{\Delta} |h|^p \, dxdy\right]^{1/p}$.

Lemma 3.24. *Fix $p > 8$. Let $\mu \in B(\mathbf{C})_1$ satisfying $\|\mu\|_{\infty} C_p < 1$. Then the canonical μ-qc mapping f^{μ} of $\hat{\mathbf{C}}$ satisfies the following integral formula:*

$$f^{\mu}(\zeta) = \zeta - \frac{1}{\pi}\iint_{\Delta} (f^{\mu})_{\bar{z}}(z) \left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta - 1}{z}\right) + \frac{(g^{\mu})_{\bar{z}}(z)}{(g^{\mu}(z))^2}\left(\frac{\zeta^2 z}{1 - z\zeta} - \frac{z\zeta}{1-z}\right) dxdy$$

for every $\zeta \in \Delta$, where $g^{\mu}(z) = \dfrac{1}{f^{\mu}(1/z)}$.

Proposition 3.25. *If μ converges to 0 in $B(C)_1$, then the canonical μ-qc mapping f^μ converges to the identity locally uniformly on* **C**.

Theorem 3.26 *Let $\{\mu(t)\}$ be a family of Beltrami coefficients depending on a parameter t. Suppose that $\|\mu(t)\|_\infty \to 0$ as $t \to 0$, and that $\mu(t)$ is differentiable at $t = 0$. This means*

$$\mu(t)(z) = tv(z) + t\varepsilon(t)(z), \qquad z \in \mathbf{C},$$

for some $v, \varepsilon(t) \in L^\infty(\mathbf{C})$ such that $\|\varepsilon(t)\|_\infty \to 0$ as $t \to 0$. Then

$$\dot{f}[v](\zeta) = \lim_{t \to 0} \frac{f^{\mu(t)}(\zeta) - \zeta}{t} = -\frac{1}{\pi}\iint_{C} \frac{v(z)\zeta(\zeta - 1)}{z(z-1)(z-\zeta)}\, dxdy$$

exists for every $\zeta \in \mathbf{C}$, and the convergence is locally uniform on **C**. [Im,Ta]

Corollary. *Let $\{\mu(t)\}$ be a family of Beltrami coefficients depending on a parameter t. Suppose that $\mu(t)$ is differentiable at $t = 0$. This means*

$$\mu(t)(z) = \mu(z) + tv(z) + t\varepsilon(t)(z), \qquad z \in \mathbf{C},$$

for some $\mu \in B(C)_1$, $v, \varepsilon(t) \in L^\infty(\mathbf{C})$ such that $\|\varepsilon(t)\|_\infty \to 0$ as $t \to 0$. Then

$$f^{\mu(t)}(\zeta) = f^\mu(\zeta) + t\dot{f}^\mu[v](\zeta) + o(|t|), \qquad \zeta \in \mathbf{C}$$

*locally uniformly on **C** as $t \to 0$, where*

$$\dot{f}^\mu[v](\zeta) = -\frac{1}{\pi}\iint_{C} \frac{v(z)\, f^\mu(\zeta)\, (f^\mu(\zeta) - 1)\, ((f^\mu)_z(z))^2}{f^\mu(z)\, (f^\mu(z) - 1)\, (f^\mu(z) - f^\mu(\zeta))}\, dxdy.$$

CHAPTER 4.

Teichmüllar Spaces

1. Analytic Construction of Teichmüllar Spaces

Teichmüllar Space of a Riemann Surface

We shall present a construction of Teichmüllar spaces by using quasiconformal mapping.

For every Riemann surface R, take a universal covering $\tilde{R}$ of R. By the Riemann mapping theorem, $\tilde{R}$ is one of $\hat{\mathbf{C}}$, $\mathbf{C}$, or H. For every homeomorphism f of R onto another Riemann surface S, there exists a homeomorphism $\tilde{f}$, a lift of f, of $\tilde{R}$ onto a universal covering $\tilde{S}$ of S. f is said to be *quasiconformal or K-qc* if $\tilde{f}$ is quasiconformal or K-qc. We consider the pair (S,f) as follows.

(S,f) and (T,g) are called *equivalent* if $g \circ f^{-1}$ is homotopic to a conformal mapping of S onto T. The set of all such equivalence classes [S,f] of (S,f) is called the *Teichmüllar space* T(R) of R.

Teichmüllar Spaces of a Fuchsian Group

Fix a non-commutative Fuchsian model Γ of R acting on H. It is noted that the set of all fixed points of elements of $\Gamma\backslash\{id\}$ contains at least 3 points.

We assume that each of 0, 1, & ∞ is a fixed point of some element of $\Gamma\backslash\{id\}$. We consider the lift $\tilde{f}$: H $\rightarrow$ H of a quasiconformal mapping f : R $\rightarrow$ S which fixes each of 0, 1, & ∞. $\tilde{f}$ is called the *canonical lift of f with respect to Γ.*

Using the canonical lift $\tilde{f}$, we have an injective homomorphism

$$\theta_{\tilde{f}} : \Gamma \rightarrow PSL(2,R)$$

which is defined by $\theta_{\tilde{f}}(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}, \gamma \in \Gamma$.

Then we have an isomorphism $\theta_{\tilde{f}}$ of Γ onto another Fuchsian group $\Gamma' := \tilde{f}\, \Gamma\, \tilde{f}^{-1}$, and $S = H/\Gamma'$.

Lemma 4.1. $[\,S, f\,] = [\,T, g\,]$ *in* $T(R) \Leftrightarrow \theta_{\tilde{f}} = \theta_{\tilde{g}}$.

$T^{\#}(\Gamma) := \{\, \theta_{\tilde{f}} : \tilde{f}$ is a canonical quasiconformal mapping of $\hat{\mathbf{C}}$ such that $\tilde{f}\, \Gamma \tilde{f}^{-1}$ is a Fuchsian group $\}$ is called the *reduced Teichmüllar space* of Γ.

Let $QC(\Gamma) = \{\, \omega : \omega$ is a canonical quasiconformal mapping of $\hat{\mathbf{C}}$ such that $\omega\, \Gamma\, \omega^{-1}$ is a Fuchsian group $\}$. ω_1 and ω_2 are *equivalent* if $\omega_1 = \omega_2$ on $\mathbf{R}$. The *Teichmüllar space* of Γ is defined by $T(\Gamma) = \{\, [\omega] : \omega \in QC(\Gamma)\, \}$.

Lemma 4.2. *Suppose that R is compact, 2 quasiconformal mappings* $f : R \to S$ *and* $g : R \to T$ *satisfy* $\theta_{\tilde{f}} = \theta_{\tilde{g}}$ *if and only if* $\tilde{f} = \tilde{g}$ *on* $\mathbf{R}$.

Remark. $T(\Gamma) \cong T^{\#}(\Gamma)$ when $R = H/\Gamma$ is a closed Riemann surface.

Proposition 4.3. *Let* Γ *be a Fuchsian model of a Riemann surface R. Then* $T(R) \cong T^{\#}(\Gamma)$. *Moreover, if R is compact, then* $T(R) \cong T(\Gamma)$.

Teichmüllar Distance

Definition. A bounded measurable function μ on H satisfying

$$\mu_{\tilde{f}} = (\mu_{\tilde{f}} \circ \gamma) \frac{\overline{\gamma'}}{\gamma'} \qquad \text{almost everywhere on H, } \gamma \in \Gamma. \qquad (*)$$

is called a *Beltrami differential* on H with respect to Γ.

Remark. If $[S,f] \in T(R)$, then the complex dilatation $\mu_{\tilde{f}}$ of the canonical lift $\tilde{f}$ of f with respect to Γ satisfies (*) become $\theta_{\tilde{f}}(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma, \gamma \in \Gamma$.

Conversely, if $\tilde{f} : H \to H$ satisfies $\mu_{\tilde{f}} = (\mu_{\tilde{f}} \circ \gamma) \frac{\overline{\gamma'}}{\gamma'}$ almost everywhere on H, $\gamma \in \Gamma$, then $\tilde{f}$ projects to a quasiconformal mapping of R onto the Riemann surface $H/\theta_{\tilde{f}}(\Gamma)$.

Definition. Let $B(H,\Gamma)$ be the set of all Beltrami differential on H with respect to Γ. Set $B(H,\Gamma)_1 = \{ \mu \in B(H,\Gamma) : \|\mu\|_\infty < 1 \}$ where μ is called the *Beltrami coefficient* on H with respect to Γ.

Similarly, we call a measurable (-1,1)-form $\mu(z)\frac{d\bar{z}}{dz}$ on R such that $\|\mu\|_\infty < \infty$ a *Beltrami differential* on R. Let B(R) be the set of all Beltrami differential on R. Set $B(R)_1 = \{ \mu \in B(R) : \|\mu\|_\infty < 1 \}$ where μ is called the *Beltrami coefficient* on R.

Remark. B(R) and $B(H,\Gamma)$ can be canonically identified with norms.

Definition. For any $p_1 = [S_1,f]$, $p_2 = [S_2,g] \in T(R)$, let $\mathcal{F}_{f,g}$ be the set of all quasiconformal mappings of S_1 onto S_2 which are homotopic to $g \circ f^{-1}$. The *Teichmüllar distance* on T(R) between p_1 and p_2 is defined by

$$d(p_1,p_2) = \inf_{h \in \mathcal{F}_{f,g}} \log K(h)$$

where K(h) is the maximal dilatation of h.

When R is compact, we define a topology on T(R) by the Teichmüllar distance.

Theorem 4.4. *T(R) is complete with respect to the Teichmüllar distance.*

Proof. Take any Cauchy sequence $\{ p_n = [S_n , f_n]\}_{n=1}^{\infty}$ in T(R) with respect to the Teichmüllar distance d. $\forall\ \varepsilon > 0$, $\exists\ N(\varepsilon)$ such that, $\forall\ n,m > N(\varepsilon)$, $\exists$ a quasiconformal mapping $f_{n\ m}$ homotopic to $f_m \circ f_n^{-1}$ and satisfying that $\|\mu_{n\ m}\|_\infty < \varepsilon$, where $\mu_{n\ m} = \mu_{f_{n\,m}}$. In particular, we find a subsequence $\{p_{n_j}\}_{n=1}^{\infty}$ and a sequence $\{f_{n_j n_{j+1}}\}_{n=1}^{\infty}$ of quasiconformal mapping such that

$$\|\mu_{n_j\ n_{j+1}}\|_\infty < \frac{1}{2^j}, \qquad j \in \mathbf{N}.$$

Now, let p_0 be the base point of T(R). Because $\{d(p_0,p_n)\}_{n=1}^{\infty}$ is a bounded sequence, we assume $K(f_n) < K$ for all n with large K. Since

$$K(f_{n_j n_{j+1}}) \le 1 + \frac{1}{2^{j-2}} \qquad \forall\, j \ge 2,$$

we have $g_j = f_{n_{j-1}n_j} \circ f_{n_{j-2}n_{j-1}} \circ \ldots f_{n_1 n_2} \circ f_{n_1}$ is a quasiconformal mapping of R onto S_{n_j}, and satisfies

$$K(g_j) \le K \prod_{r=1}^{j-1} (1 + 4 \times 2^{-r}).$$

Thus $\{K(g_j)\}_{n=1}^{\infty}$ is a bounded sequence. Set $K_1 = \sup_j \{K(g_j)\}$.

Let $\tilde{g}_j$ be the canonical lift of g_j with respect to Γ $\forall$ j. Then $\mu_j = \mu_{\tilde{g}_j} \in B(H,\Gamma)_1$, and $\|\mu_j\|_\infty \le k_1 = \frac{1 - K_1}{1 + K_1}$. Also we have

$$\frac{1}{2}\|\mu_j - \mu_{j+1}\|_\infty \le \left\|\frac{\mu_j - \mu_{j+1}}{1 - \overline{\mu_j}\,\mu_{j+1}}\right\|_\infty = \|\mu_{n_j\, n_{j+1}}\|_\infty < \frac{1}{2^j} \qquad \forall\, j.$$

In particular, $\{\mu_j\}_\infty$ is a Cauchy sequence in B(H,Γ). Hence, $\mu_j \to \mu$ in B(H,Γ) as n $\to \infty$, and satisfies $\|\mu\|_\infty \le k_1$.

Let $\tilde{f}$ be the canonical μ-qc mapping of H which belongs to QC(Γ). Let p = [S,f] ∈ T(R) determined by $\theta_{\tilde{f}}$. Since

$$\tanh\left[\frac{d(p_{n_j},p)}{2}\right] \le \left\|\frac{\mu_j - \mu}{1 - \overline{\mu_j}\,\mu}\right\|_\infty \le \frac{1}{1-(k_1)^2}\|\mu_j - \mu\|_\infty,$$

$p_{n_j} \to p$ as $j \to \infty$. Thus $p_n \to p$.

□

Fix $[R_1, f_1] \in T(R)$. By setting

$$[f_1]_*([S, f]) = [S, f \circ f_1^{-1}], \qquad [S, f] \in T(R),$$

we define a surjective mapping $[f_1]_* : T(R) \to T(R_1)$ with base point $[R_1, id]$.

Proposition 4.5. *$[f_1]_*: T(R) \to T(R_1)$ is an isometrical homeomorphism with respect to the Teichmüller distances. In particular, $T(R_1)$ is homeomorphic to $T(R)$.*

Definition 4.6. By proposition 4.5, T(R) are mutually homeomorphic for all closed Riemann surfaces R of genus $g \geq 2$. The Teichmüllar spaces for such R is independent of the base point. We denote such a space by T_g, and call it the *Teichmüllar space of genus g*.

2. Teichmüller Mappings and Teichmüller's Theorems

Holomorphic Quadratic Differentials

Let (U_j, z_j) be a coordinate neighborhood of a Riemann surface R. A *holomorphic quadratic differentials* on R is defined by a family $\{\varphi_j\}$ of holomorphic functions φ_j on $z_j(U_j)$ satisfying

$$\varphi_k(z_k) = \varphi_j(z_j) \left(\frac{dz_j}{dz_k}\right)^2 \qquad \text{on } U_j \cap U_j.$$

We write

$$\varphi = \varphi(z)\, dz^2.$$

Analogously, a holomorphic function $\varphi(z)$ on H will be called a *holomorphic automorphic form* with respect to Γ if

$$\varphi(\gamma(z))\, \gamma'(z)^2 = \varphi(z), \quad z \in H, \gamma \in \Gamma.$$

Definition. Let $A_2(R)$ be the complex vector space of all holomorphic quadratic differentials on R. Let $A_2(H,\Gamma)$ be the complex vector space of all holomorphic automorphic form with respect to Γ.

Remark. $A_2(R)$ and $A_2(H,\Gamma)$ are naturally identified.

Teichmüller Mappings

Definition. Let $A_2(R)_1 = \{ \varphi \in A_2(R) : \|\varphi\|_1 := 2\iint_R |\varphi(z)| \, dxdy < 1 \}$.

Let $\varphi \in A_2(R)_1$. By proposition 3.22, we can find a quasiconformal mapping f such that

$$\mu_f = \|\varphi\|_1 \frac{\overline{\varphi}}{|\varphi|}.$$

Further, $[f(R),f]$ is well-defined.

Definition. f is called a *Teichmüller mapping* for φ.

We shall show that the mapping $\mathcal{T} : A_2(R)_1 \to T(R)$ is defined by

$$\mathcal{T}(\varphi) = [f(R),f], \qquad \varphi \in A_2(R)_1,$$

is a surjective homeomorphism in Theorem 4.14 where $f : R \to f(R)$ is a Teichmüller mapping for $\varphi \neq 0$, and $f = id$ for $\varphi = 0$.

Remark. We shall identify the following Teichmüller spaces $T(R)$, T_g, T_g^{marked} (by using marked Riemann surfaces of genus g) and $T(R)^{diffeo}$(by using orientation-preserving diffeomorphisms) by the following mappings.

Fix a marking $\Sigma = \{[A_j,B_j]\}_{j=1}^{g}$ on R, A mapping $\Phi_\Sigma : T(R) \to T_g^{marked}$ is given by

$$\Phi_\Sigma([S,f]) = [S,f_*(\Sigma)], \quad [S,f] \in T(R),$$

where $f_*(\Sigma) = \{f_*([A_j]),f_*([B_j])\}_{j=1}^{g}$. The bijective mapping of $T(R)$ into T_g is given by the Definition 4.6. Moreover, $\Phi_\Sigma : T(R)^{diffeo} \to T_g^{marked}$ is shown in Theorem 1.1.

Lemma 4.7. *$\phi_\Sigma : T(R) \to T_g^{old}$ and $\mathcal{F}_g o \phi_\Sigma : T(R) \to F_g$ are bijective. In particular, $F_g = \mathcal{F}_g o \phi_\Sigma(T(R))$.*

Teichmüller's Theorems

We start with Teichmüller's uniqueness theorem. The sketch of proof of the theorem will be contained in section 3.

Teichmüller's uniqueness theorem 4.8. *Let f be a Teichmüller mapping for $\varphi \in A_2(R)_1$, and let $\mathcal{T}(\varphi) = [S,f]$. Then every quasiconformal mapping $g : R \to S$ which is homotopic to f satisfies*

$$\|\mu_g\|_\infty \geq \|\mu_f\|_\infty .$$

Moreover, $\|\mu_g\|_\infty = \|\mu_f\|_\infty \Leftrightarrow g = f$.

Assuming the Teichmüller's uniqueness theorem, we can have the following results.

Corollary. *$\mathcal{T}$ and $\tilde{\mathcal{T}}$ are injective.*

Lemma 4.9. *$\tilde{\mathcal{T}}(A_2(R)_1)$ is open in F_g, and $\tilde{\mathcal{T}}$ is a homeomorphism onto its image.*

Proof. It can be checked that $\tilde{\mathcal{T}} := \mathcal{F}_g \circ \phi_\Sigma \circ \mathcal{T} : A_2(R)_1 \to F_g$ is continuous. Since $\tilde{\mathcal{T}}$ is injective and $A_2(R)_1$ is homeomorphic to $\mathbf{R}^{6g-6}$ by Riemann-Roch theorem, Brouwers' theorem on invariance domain gives the assertion.

□

Remark. The Riemann-Roch theorem: On compact Riemann surface of genus g, the space of holomorphic quadratic differentials has dimension 1 if g=1 and 3g-3 if $g \geq 2$.

Lemma 4.10. *$\mathcal{T} : A_2(R)_1 \to T(R)$ is a homeomorphism onto its image.*

Proof. By the Corollary and Theorem 4.8, $\mathcal{T}^{-1} = \tilde{\mathcal{T}}^{-1} \circ (\mathcal{F}_g \circ \phi_\Sigma)$ is well-defined on E $= \mathcal{T}(A_2(R)_1)$. It can be checked that $\mathcal{F}_g \circ \phi_\Sigma : T(R) \to F_g$ is continuous. So $\mathcal{T}^{-1}$ is continuous on E. It suffices to show that $\mathcal{T}$ is continuous on E.

Fix $\varphi \in A_2(R)_1$, and set $p = \mathcal{T}(\varphi) = [R_1, f_1]$. Consider the surjective isometry $[f_1]_* : T(R) \to T(R_1)$. We define a mapping $\tilde{\mathcal{T}}_1 = \mathcal{F}_g \circ \phi_\Sigma \circ [f_1]_*^{-1} \circ \mathcal{T}_1$: $A_2(R)_1 \to F_g$ as $\tilde{\mathcal{T}}$.

Since $\tilde{\mathcal{T}}(\varphi) = \tilde{\mathcal{T}}_1(0)$, $\tilde{\mathcal{T}}_1^{-1} \circ \tilde{\mathcal{T}} : A_2(R)_1 \to A_2(R_1)_1$ is well-defined in some neighborhood of φ, and is a homeomorphism onto its image. Hence, $\mathcal{T}$ is continuous at $\varphi \Leftrightarrow \mathcal{T}_1$ is continuous at 0.

Take any sequence $\{\psi_n\}_{n=1}^{\infty}$ in $A_2(R)_1$ such that $\|\psi_n\|_1 \to 0$ as $n \to \infty$. Since the maximal dilatation of a Teichmüller mapping for ψ_n is equal to

$$\frac{1 + \|\psi_n\|_1}{1 - \|\psi_n\|_1} \qquad \forall\, n,$$

we have

$$d(\mathcal{T}_1(0), \mathcal{T}_1(\psi_n)) \le \log \frac{1 + \|\psi_n\|_1}{1 - \|\psi_n\|_1} \to 0 \qquad \text{as } n \to \infty.$$

Thus $\mathcal{T}_1$ is continuous at 0. Since φ is arbitrary, $\mathcal{T}$ is continuous .

□

It can be checked that T(R) and F_g are connected. We use it to show the following lemma.

Lemma 4.12. *$\mathcal{T}$ and $\tilde{\mathcal{T}}$ are surjective.*

Proof. It suffices to show the assertion for $\mathcal{T}$.

Since $E = (\mathcal{F}_g \circ \phi_\Sigma)^{-1}(\tilde{\mathcal{T}}(A_2(R)_1))$ is open in the connected T(R), we have to show ∂E in T(R) is empty.

Suppose $\partial E \neq \phi$. Take $[S,f] \in \partial E$. Then $\exists$ a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $A_2(R)_1$ such that $T(\varphi_n) \to [S,f]$, and $\|\varphi_n\|_1 \to 1$ as $n \to \infty$. Let f_n be a Teichmüller mapping for φ_n. Set $T(\varphi_n) = [S_n, f_n]$. There exists a quasiconformal mapping h_n : $S_n \to S$ which is homotopic to $f \circ f_n^{-1}$ $\forall$ n such that $\|\mu_{h_n}\|_\infty \to 0$ as $n \to \infty$. In particular, we have

$$\|\mu_{g_n}\|_\infty \le k \qquad \text{for some } k < 1$$

where $g_n = h_n^{-1} \circ f$ and $n \in \mathbf{N}$.

On the other hand, since g_n is homotopic to f_n, Teichmüller's uniqueness theorem implies that

$$\|\mu_{g_n}\|_\infty \geq \|\mu_{f_n}\|_\infty = \|\varphi_n\|_1 \to 1 \text{ as } n \to \infty.$$

This is a contradiction. Hence ∂E is empty.

□

By using lemma 4.12, for every $[S,f] \in T(R)$, there exists $\varphi \in A_2(R)_1$ such that $\mathcal{T}(\varphi) = [S,f]$. Thus for every qc mapping $f : R \to S$, there exists $\varphi \in A_2(R)_1$ such that $\mu_f = \|\varphi\|_1 \frac{\overline{\varphi}}{|\varphi|}$ can be solved. Thus we have the Teichmüller mapping g which is homotopic to f. Then we obtain the Teichmüller's existence theorem.

Teichmüller's existence theorem 4.13. *For every quasiconformal mapping $f : R \to S$, there exists a Teichmüller mapping homotopic to f.*

Summarize the above results, we have the following theorem.

Theorem 4.14. *$\mathcal{T} : A_2(R)_1 \to T(R)$ is a surjective homeomorphism. In particular, $T(R)$ is homeomorphic to $A_2(R)_1$, and hence to $\mathbf{R}^{6g-6}$.*

Corollary. *$T(R)$, T_g, $T(R)^{old}$, T_g^{old}, F_g, and $\mathbf{R}^{6g-6}$ are mutually homeomorphic to each other.*

3. Teichmüller's Uniqueness Theorem

Let $\varphi \in A_2(R)_1\backslash\{0\}$. If $p_0 \in R$ is not a zero of φ, then $\varphi^{1/2} = \varphi(z)^{1/2}dz$ has a single valued holomorphic branch in some neighborhood U of p_0, and the conformal mapping defined by

$$\zeta(p) = \int_{p_0}^{p} \varphi^{1/2}, \qquad p \in U$$

which is called a *φ-coordinate* around p_0.

At a zero $p_0$of φ of order $m \geq 1$, it can be shown that φ is written in the form

$$\varphi = z^m \, dz^2$$

for a local coordinate z on some neighborhood U of p_0. Hence, we see that

$$\zeta(p) = \frac{2}{m+2} z^{(m+2)/2}$$

cannot be single-valued in any neighborhood of p_0. We may consider that ζ mappingss $\{ z \in \mathbf{C} : 0 < \arg z < 2\pi, 0 < |z| < r \}$ conformally onto $\{ \zeta \in \mathbf{C} : 0 < \arg z < (m+2)\pi, 0 < |z| < \frac{2r^{(m+2)/2}}{m+2} \}$ spread over the ζ-plane. We also call ζ a *φ-coordinate* around p_0.

Now, we consider the lift $\tilde{\varphi} \in A_2(H,\Gamma)$ of φ on H with respect to a Fuchsian model Γ of R.

Definition. For every piecewise smooth curve C on H, the length

$$|C|_{\tilde{\varphi}} = \int_C |\tilde{\varphi}(z)^{1/2}|dz|.$$

is called the *$\tilde{\varphi}$-length* of C. For any $z_1, z_2 \in H$, let $\mathcal{L}_{z_1,z_2}$ be the set of all piecewise smooth curves connecting z_1 and z_2 in H. The distance

$$d_{\tilde{\varphi}}(z_1, z_2) = \inf_{C \in \mathcal{L}_{z_1 z_2}} |C|_{\tilde{\varphi}}$$

is called the *$\tilde{\varphi}$-distance* between z_1 and z_2. An element C_0 of $\mathcal{L}_{z_1,z_2}$ is called a *$\tilde{\varphi}$-geodesic* between z_1 and z_2 if

$$|C_0|_{\tilde{\varphi}} = d_{\tilde{\varphi}}(z_1, z_2).$$

A closed arc L on H is called a *$\tilde{\varphi}$-segment* if, for every interior point p of L, L is mapped by a $\tilde{\varphi}$-coordinate at p to a segment. Every $\tilde{\varphi}$-geodesic connecting 2 points of H consists of a finite number of $\tilde{\varphi}$-segments whose end points are either z_1, or z_2, or zeros of $\tilde{\varphi}$. Moreover, two such $\tilde{\varphi}$-segments make an angle not less than $\frac{2\pi}{m+2}$ at a zero of order $m > 0$.

It can be checked that H is complete with respect to $\tilde{\varphi}$-distance. In particular, for any two points of H, there exists a $\tilde{\varphi}$-geodesic connecting them.

Proposition 4.15. *For z_1, $z_2 \in H$, the $\tilde{\varphi}$-geodesic connecting z_1 and z_2 is unique.*

Corollary. *A $\tilde{\varphi}$-segment is the unique $\tilde{\varphi}$-geodesic connecting its end points.*

Definition. The $\tilde{\varphi}$-length $|L|_{\varphi}$ of a curve L on R is defined by

$$|L|_{\varphi} = \int_L |\varphi|^{1/2}.$$

The projection of $\tilde{\varphi}$-segment to R is called a *φ-segment.*

Some propositions are prepared for the sketch of proof of the uniqueness theorem.

Lemma 4.16. (Teichmüller) *Let $h : R \to R$ be a qc mapping homotopic to id. Then there exists a constant $M(R,h,\varphi) > 0$ such that*

$$|h(L)|_{\varphi} \geq |L|_{\varphi} - M.$$

Proposition 4.17. *Let f be an affine qc mapping of $R = [0,r] \times [0,1]$ to $S = [0,s] \times [0,1]$ defined by $f(z) = Kx + iy$ which is K-qc on int(R) where $K = \frac{s}{r}$ and $k = \frac{K-1}{K+1}$.*

Then for every homeomorphism $f_1 : R \to S$ which is $\frac{1+k_1}{1-k_1}$-qc on int(R), and

$$f(0) = 0, \qquad f(r) = s, \qquad f(r+i) = s+i, \qquad \text{and} \qquad f(i) = i,$$

it follows that $k_1 \geq k$. Moreover, $k_1 = k \Leftrightarrow f_1 = f$.

Proposition 4.18. *Fix $\varphi \in A_2(R)_1$, let $f : R \to S$ be a Teichmüller mapping for φ and $k = \|\varphi\|_1$. Then there exists a unique holomorphic quadratic differential ψ on S satisfying the following conditions:*

1. *If p is a zero of φ of order m, then $f(p)$ is a zero of ψ of order m.*

2. *Let $p \in R$ which is not is a zero of φ, and ζ be a φ-coordinate around p. Then there a ψ-coordinate w at $f(p)$ such that*

$$w \circ f = \frac{\zeta + k\bar{\zeta}}{1 - k}.$$

φ and ψ are called the *initial differential of f* and the *terminal differential of f* respectively.

Sketch of Proof of Teichmüller Uniqueness Theorem

Teichmüller's uniqueness theorem. *Let f be a Teichmüller mapping for $\varphi \in A_2(R)_1$, and let $\mathcal{T}(\varphi) = [S,f]$. Then every quasiconformal mapping $f_1 : R \to S$ which is homotopic to f satisfies*

$$\|\mu_{f_1}\|_\infty \geq \|\mu_f\|_\infty .$$

Moreover, $\|\mu_{f_1}\|_\infty = \|\mu_f\|_\infty \Leftrightarrow f_1 = f$.

Skerch of proof. Let ψ be the terminal differential of f . For every p $\in$ R which is not a zero of φ, take a φ-coordinate $\zeta = \xi + i\eta$ around p, and a ψ-coordinate $\omega = \sigma + i\tau$ around q = f(p). Applying Teichmüller lemma, we can show that

$$\iint_S \lambda(g,q)\, d\sigma d\tau \geq \iint_S d\sigma d\tau$$

where $\lambda(g,q) = \left|\frac{\partial(\omega \circ g \circ \omega^{-1})}{\partial\sigma}\right|(0)$ and $g = f_1 \circ f^{-1}$. By applying Schwarz inequality, we obtain

$$\iint_S \lambda(g,q)^2 \, d\sigma d\tau \geq \iint_S d\sigma d\tau.$$

Set $k = \|\mu_{f_1}\|_\infty$, $k_1 = \|\mu_f\|_\infty$, $\hat{f}_1 = \omega \circ f_1 \circ \zeta^{-1}$ and $J(f_1)(p) = |\hat{f}_{1\zeta}|(0)^2 - |\hat{f}_{1\bar{\zeta}}|(0)^2$. Then we have

$$\lambda(f_1,q)^2 \leq K_1 \, J(f_1)(p) \quad \text{almost everywhere on R}$$

where $K_1 = \dfrac{1 + k_1}{1 - k_1}$, $K = \dfrac{1 + k}{1 - k}$.

Since $K \, \lambda(g,f(p)) = \lambda(f_1,p)$ for almost every $p \in R$ and $d\sigma d\tau = K \, d\xi d\eta$, we have

$$\begin{aligned}\iint_S d\sigma d\tau &\leq \iint_R \left(\frac{\lambda(g'q)}{K}\right)^2 K d\xi d\eta \\ &\leq \frac{K_1}{K} \iint_R J(f_1)(p) \, d\xi d\eta \\ &\leq \frac{K_1}{K} \iint_S d\sigma d\tau .\end{aligned}$$

Thus $K \leq K_1 \Leftrightarrow k \leq k_1$.

Now we consider the equality. If $k_1 = k$, then

$$|\hat{f}_{1\zeta} + \hat{f}_{1\bar{\xi}}|(0) = |\hat{f}_{1\zeta}|(0) + |\hat{f}_{1\bar{\xi}}|(0)$$

and

$$|\hat{f}_{1\zeta}|(0) = k \, |\hat{f}_{1\bar{\xi}}|(0) \qquad \text{almost everywhere on R.}$$

This implies that $\mu_{f_1} = k \dfrac{\varphi}{|\varphi|}$. So g is conformal on R. Since g is homotopic to id, the canonical lift of g on H is coincident with id by lemma 4.2. Thus $f_1 = f$.

□

CHAPTER 5

Complex Analytic Theory of Teichmüller Spaces

1. Bers' Embedding and the Complex Structure of Teichmüller Space

Simultaneous Uniformization

We shall represent the Teichmüller space $T(\Gamma)$ by quasiconformal mappings of the Riemann surface $\hat{\mathbf{C}}$ which are conformal on the lower half-plane H^*.

For $\mu \in B(H, \Gamma)_1$, set

$$\hat{\mu}(z) = \begin{cases} \mu(z) & z \in H \\ 0 & z \in \mathbf{C} \setminus H. \end{cases}$$

From Theorem 3.19, there exists uniquely a canonical $\hat{\mu}$-qc mapping w_μ of $\hat{\mathbf{C}}$.

For any $\gamma \in \Gamma$, let $\chi_\mu(\gamma) = w_\mu \circ \gamma \circ w_\mu^{-1} \in \mathrm{Aut}(\hat{\mathbf{C}})$. Thus a subgroup $\Gamma_\mu = \{\chi_\mu(\gamma) : \gamma \in \Gamma\}$ of $\mathrm{Aut}(\hat{\mathbf{C}})$ acts properly discontinuously on $H_\mu \overset{\text{def}}{=} w_\mu(H)$ and $H_\mu^* \overset{\text{def}}{=} w_\mu(H^*)$. Γ_μ is a quasi-Fuchsian group, that is, a discrete subgroup of $PSL(2,\mathbf{C})$ which leaves a directed simple closed curve in $\hat{\mathbf{C}}$ fixed. It is checked that every element of $\Gamma_\mu \setminus \{id\}$ has no fixed points on both H_μ and H_μ^*.

The quasiconformal mapping w_μ induces a quasiconformal mapping of $R = H/\Gamma$ to $R_\mu = H_\mu/\Gamma_\mu$ and a biholomorphic mapping of $R^* = H^*/\Gamma$ to H_μ^*/Γ_μ, R_μ and R^* are uniformized simultaneously by a single quasi-Fuchsian group Γ_μ. This is referred to as *Bers' simultaneous uniformization.*

In particular, for any closed Riemann surfaces R and S of genus g, we find a quasi-Fuchsian group Γ_μ which uniformizes simultaneously R and S.

Lemma 5.1. *For $\mu, \nu \in B(H, \Gamma)_1$, $w^\mu = w^\nu$ on $\mathbf{R} \Leftrightarrow w_\mu = w_\nu$ on H^*.*

Definition. w_μ and w_v are said to be *equivalent* if $w_\mu = w_v$ on H^*. Let $T_\beta(\Gamma)$ be the set of all equivalence classes $[w_\mu]$ of w_μ. Since $T_\beta(\Gamma)$ can be identified by $T(\Gamma)$ as topological spaces, $T_\beta(\Gamma)$ is also called the *Teichmüller space of Γ*.

A mapping $\beta : B(H, \Gamma)_1 \to T_\beta(\Gamma)$ is defined by $\beta(\mu) = [w_\mu]$. It is checked that β is a continuous surjection.

Schwarzian Derivative

For any conformal mapping f on a domain in **C**, a *Schwarzian derivative* of f is defined by

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left[\frac{f''(z)}{f'(z)}\right]^2 .$$

Remark. There are some equalities of Schwarzian derivative {f , z}

$$\{f, z\} = \{c f + d\ , z\} = \{\tfrac{1}{f}, z\} \qquad \text{where c and d are constants.}$$

Lemma 5.2. If f and g are conformal mapping of D and f(D), respectively, then

$$\{ g \circ f , z \} = \{ g, f(z) \} f'(z)^2 + \{f , z\}, \qquad z \in D.$$

Moreover, a conformal mapping of D is a Möbius transformation $\Leftrightarrow$ $\{f , z\} = 0$ on D.

Bers' Embedding and the Complex Structure of Teichmüller Space

For any $\mu \in B(H, \Gamma)_1$, set

$$\varphi_\mu = \{w_\mu, z\}, \qquad z \in H^*.$$

A mapping $\mathbf{B} : T_\beta(\Gamma) \to A_2(H^*,\Gamma)$ is defined by

$$\mathbf{B}([w_\mu]) = \varphi_\mu.$$

Lemma 5.3. *If $\gamma \in \Gamma$, then*

$$\varphi_\mu(\gamma(z))\,\gamma'(z)^2 = \varphi_\mu(z), \qquad z \in H^*.$$

Moreover, for any $\mu, \nu \in B(H, \Gamma)_1$, $[w_\mu] = [w_\nu]$ in $T_\beta(\Gamma) \Leftrightarrow \varphi_\mu = \varphi_\nu$ on H^*.

B is well-defined and injective, and that is called *Bers' embedding*. The mapping $\Phi : B(H, \Gamma)_1 \to A_2(H^*,\Gamma)$ given by $\Phi(\mu) = \mathbf{B} \circ \beta(\mu)$ is called *Bers' projection.*

Now, we introduce the hyperbolic L^∞-norm on $A_2(H^*,\Gamma)$ by using Poincaré metric $ds_{H^*}{}^2$. Every element $\varphi \in A_2(H^*,\Gamma)$ satisfies

$$(\operatorname{Im} \gamma(z))^2 |\varphi(\gamma(z))| = (\operatorname{Im} z)^2 |\varphi(z)|, \quad z \in H^*, \gamma \in \Gamma.$$

So $(\operatorname{Im} z)^2|\varphi(z)|$ is regarded as a function on R^*. The *hyperbolic L^∞-norm* of φ in $A_2(H^*,\Gamma)$ is defined by

$$\|\varphi\|_\infty = \sup_{z \in H^*} (\operatorname{Im} z)^2 |\varphi(z)| < \infty.$$

Proposition 5.4. *Φ and **B** are continuous.*

The Brouwer's theorem on invariance of domains implies that the image $T_B(\Gamma) = \mathbf{B}(T_\beta(\Gamma))$ is a domain in $A_2(H^*,\Gamma)$,and $\mathbf{B} : T_\beta(\Gamma) \to T_B(\Gamma)$ is a homeomorphism. Since $A_2(H^*,\Gamma)$ is a 3g-3 - dimensional complex vector space, $T_B(\Gamma)$ inherits the complex manifold structure of $A_2(H^*,\Gamma)$. Under the identification with $T_B(\Gamma)$, $T(\Gamma)$, $T_\beta(\Gamma)$ and $T(R)$ are considered as 3g-3 -dimensional complex manifold, where $R = H/\Gamma$. Henceforth, $T_B(\Gamma)$, $T(\Gamma)$ and $T_\beta(\Gamma)$ will be called the *Teichmüller space* of Γ.

Boundness of $T_B(\Gamma)$

Lemma 5.6. (Nehari and Kraus) *Every univalent function satisfies*

$$\| \{f,z\} \|_\infty \leq \frac{3}{2}.$$

Proof. Consider a univalent function on $\Delta^* := \hat{\mathbb{C}} \setminus \bar{\Delta}$. We can check that if F is given by

$$F(w) = w + b_0 + \frac{b_1}{w} + \dots, \qquad (*)$$

then $|b_1| < 1$ by applying Bieberbach's area theorem.

Differentiating the series (*), we obtain

$$\{F,w\} = -\frac{6b_1}{w^4} + \sum_{n=5}^{\infty} \frac{c_n}{w^n}, \quad w \in \Delta^* \setminus \{\infty\}.$$

Then we have

$$\lim_{w \to \infty} |w^4 \{F,w\}| = 6\,|b_1| \le 6.$$

Let f be an univalent function on H*. For $z_0 = x_0 + iy_0$, suppose $f(z_0) \neq \infty$. Taking $T : H^* \to \Delta^*$ defined by

$$T(z) = \frac{z - \overline{z_0}}{z - z_0},$$

we get

$$F(w) = \frac{2iy_0 f'(z_0)}{f(T^{-1}(w)) - f(z_0)}, \qquad w \in \Delta^*.$$

Then F is a univalent function on Δ^*, and has an expression as (*). We have $\{f,z\} = \{F,T(z)\}\, T'(z)^2$ on H*. Then

$$|\{f,z_0\}| = \lim_{z \to z_0} |\{F,T(z)\}\, T'(z)^2|$$
$$\le \frac{3}{2y_0^2}.$$

For $f(z_0) = \infty$. By the relation $\{f,z_0\} = \{1/f,z_0\}$, we also have $|\{f,z_0\}| \le \dfrac{3}{2y_0^2}$.

Thus $\| \{f,z\} \|_\infty \le \frac{3}{2}$.

□

By applying the lemma, we get

Theorem 5.7. *$T_B(\Gamma)$ is contained in the open ball of radius $\frac{3}{2}$ centered at 0 in $A_2(H^*,\Gamma)$.*

2. Invariance of Complex Structure of Teichmüller Space

We shall show that the complex structure of $T(\Gamma)$ is independent of the Fuchsian model Γ of a closed Riemann surface of genus $g \geq 2$.

Local Inverse of Bers' Embedding

For any $\varphi \in A_2(H^*,\Gamma)$, set

$$\mu_\varphi(z) = -2\,(\mathrm{Im}\, z)^2\, \varphi(\overline{z}), \qquad z \in H,$$

then $\mu_\varphi \in B(H,\Gamma)$ is called the *Bers' Beltrami differential* constructed from φ. Let $V = \{\, \varphi \in A_2(H^*,\Gamma) : \|\varphi\|_\infty < \frac{1}{2} \}$. Then $\mu_\varphi \in B(H,\Gamma)_1$.

A mapping $\psi : V \rightarrow T_\beta(\Gamma)$ is defined by

$$\psi(\varphi) = [w_{\mu_\varphi}]$$

which is continuous from proposition 5.2.

Theorem 5.8. *Under the preceding situation, $U = \psi(V)$ is an open neighborhood of the base point in $T_\beta(\Gamma)$, and $\psi : V \rightarrow U$ is the inverse of $\boldsymbol{B} : U \rightarrow V$.*

The theorem is a result of the following theorem due to Ahlfors and Weil. [Ah,We]

Theorem 5.9. (Ahlfors and Weill) *For every $\varphi \in V$, the Bers' Beltrami differential μ_φ constructed from φ satisfies*

$$B([w_{\mu\varphi}]) = \varphi.$$

Corollary. *For every $\varphi \in V$, the $\mu \in B(H,\Gamma)_1$ such that w_μ is real-analytic on H and $B^{-1}(\varphi) = [w_\mu]$. Moreover, every $[S,f] \in T(H/\Gamma)$ is represented by a real-analytic qc mapping $g : H/\Gamma \to S$.*

Differentiation of Bers' Projection

We want to define the derivative $\dot{\phi}_\mu[v]$ of φ in the direction $v \in B(H,\Gamma)$ at $\mu \in B(H,\Gamma)_1$.

Let $\mu_t \in B(H,\Gamma)_1$ in a neighborhood of the origin such that $\mu_t = \mu + tv + t\varepsilon(t)$, where $v \in B(H,\Gamma)$, and $\|\varepsilon(t)\|_\infty \to 0$ as $t \to 0$. Then $\dot{\phi}_\mu[v]$ is defined by

$$\dot{\phi}_\mu[v] = \lim_{t \to 0} \frac{\phi(\mu_t) - \phi(\mu)}{t},$$

where the convergence is norm convergence with respect to $L^\bullet$-norm. The existence and integral representation of $\dot{\phi}_\mu[v]$ is given as follows.

Theorem 5.10. *For every $v \in B(H,\Gamma)$, $\dot{\phi}_0[v]$ exists and is given by*

$$\dot{\phi}_0[v](z) = -\frac{6}{\pi}\iint_H \frac{v(\zeta)}{(\zeta - z)^4}\, d\xi d\eta, \qquad z \in H^*.$$

Proof. By Theorem 3.26, we have

$$w_{\mu_t}(z) = z + t\dot{w}[v](z) + o(t)$$

uniformly on compact sets of **C** as $t \to 0$, where

$$\dot{w}[v](z) = -\frac{1}{\pi}\iint_H \frac{v(\zeta)\, z(z-1)}{\zeta(\zeta-1)(\zeta - z)}\, d\xi d\eta.$$

Since $w_{\mu_t}(z)$ is holomorphic on H^*, from Weierstass' theorem on double series, we get w_{μ_t}', w_{μ_t}'' and w_{μ_t}''' uniformly on compact set of H^* as $t \to 0$. Since H^*/Γ is compact, $\dot{\phi}_0[v]$ exists and is equal to w_{μ_t}'''.

□

Theorem 5.11. *For every $v \in B(H,\Gamma)$ and $\mu \in B(H,\Gamma)_1$, $\dot{\phi}_\mu[v]$ exists and given by*

$$\dot{\phi}_\mu[v] = \left[-\frac{6}{\pi} \iint_H \frac{v(\zeta)\,((w_\mu)_\zeta(\zeta))^2}{(w_\mu(\zeta) - w_\mu(z))^4}\, d\xi d\eta \right] w_\mu'(z)^2 , \qquad z \in H^*.$$

Proof. Set $f = w_\mu$, $g_t = w_{\mu_t} \circ f^{-1}$ and $\lambda_t = \mu_{g_t}$. Then we have

$$\lambda_t(\zeta) = \frac{f_z}{\overline{f_z}} \frac{\mu_t - \mu}{1 - \overline{\mu}\, \mu_g} \circ f^{-1}(\zeta) \ , \ \zeta \in f(H).$$

Putting

$$\lambda(\zeta) = \frac{f_z}{\overline{f_z}} \frac{v}{1 - |\mu|^2} \circ f^{-1}(\zeta) \ ,$$

we have

$$\lambda_t = t\lambda + t\delta(t) \quad \text{on } f(H),$$

where $\|\delta(t)\|_\infty \to 0$ as $t \to 0$.

Besides, $\qquad \phi(\mu_t)(z) = \{g_t \circ f, z\} f\,'(z)^2 + \phi(\mu)(z),$

we obtain

$$\dot{\phi}_\mu[v] = \lim_{t \to 0} \frac{\{g_t \circ f, z\}}{t} f\,'(z)^2, \qquad z \in H^*.$$

By using the same argument as in the proof of theorem 5.10, $\dot{\phi}_\mu[v]$ exists and is given by

$$\dot{\phi}_\mu[v] = \left[-\frac{6}{\pi} \iint_{f(H)} \frac{\lambda(\zeta)}{(\zeta - f(z))^4}\, d\xi d\eta \right] f\,'(z)^2 , \qquad z \in H^*.$$

By substituting $f(\zeta)$ for ζ in the integral, we obtain the result.

□

Invariance of Complex Structure of $T(\Gamma)$

We shall show that the complex structure of $T(\Gamma)$ is independent of Γ. Let Γ_1 be a Fuchsian model of a Riemann surface R_1 of genus of g such that each of points 0, 1,and ∞ is fixed by an element in $\Gamma_1 \setminus \{id\}$. Let ω be a lift of a qc mapping $f_1 : R \to R_1$, and suppose that $\Gamma_1 = \omega\Gamma\omega^{-1}$. By identifying $T(R)$, $T(R_1)$ with $T(\Gamma)$, $T(\Gamma_1)$ respectively, $[f_1]_* : T(R) \to T(R_1)$ induces a homeomorphsim $[\omega]_* : T(\Gamma) \to T(\Gamma_1)$:

$$[\omega]_* [w^\mu] = [\alpha \circ w^\mu \circ \omega^{-1}],$$

where a is a real Möbius transformation such that $\alpha \circ w^\mu \circ \omega^{-1}$ fixes each of 0, 1, and ∞. A homeomorphism $<\omega>_* : T_\beta(\Gamma) \to T_\beta(\Gamma_1)$ is defined by

$$<\omega>_* ([w_\mu]) = [w_\nu].$$

Lemma 5.12. The Jacobian of an injective holomorphic mapping $F = (F_1, \dots F_n)$ of a domain D in $\mathbf{C}^n$ into $\mathbf{C}^n$ is nowhere vanishing on D.

The lemma will be used in the following theorem. Let $\mathbf{B}_1 : T_\beta(\Gamma_1) \to A_2(H,\Gamma_1)$ be the Bers' embedding of $T_\beta(\Gamma_1)$. We have

Theorem 5.13. *The following mappings are biholomorphic:*

$$[f_1]_* : T(R) \to T(R_1)\,,$$

$$[\omega]_* : T(\Gamma) \to T(\Gamma_1),$$

$$<\omega>_* : T_\beta(\Gamma) \to T_\beta(\Gamma_1),$$

$$F = \mathbf{B}_1 \circ <\omega>_* \circ \mathbf{B}^{-1} : T_B(\Gamma) \to T_B(\Gamma_1).$$

Proof. We have to show that F is holomorphic in a neighborhood of $\mathbf{B}([w_\mu]) \in T_B(\Gamma)$. Let $\Gamma^\mu = w^\mu\Gamma(w^\mu)^{-1}$ and $\mathbf{B}_\mu$ be Bers' embedding of $T_\beta(\Gamma^\mu)$. If

$$F_1 = \mathbf{B} \circ <(w^\mu)^{-1}> \circ \mathbf{B}_\mu^{-1} : T_B(\Gamma^\mu) \to T_B(\Gamma),$$

$$F_2 = \mathbf{B}_1 \circ <\omega \circ (w^\mu)^{-1}> \circ \mathbf{B}_\mu^{-1} : T_B(\Gamma^\mu) \to T_B(\Gamma_1)$$

are biholomorphic in a neighborhood of the base point of $T_B(\Gamma^\mu)$, $F = F_2 \circ F_1^{-1}$ is biholomorphic in a neighborhood of $\mathbf{B}([w_\mu]) \in T_B(\Gamma)$.

It is sufficient to show that F is holomorphic in a neighborhood of the base point of $T_B(\Gamma)$. Let V be a neighborhood of the base point of $T_B(\Gamma)$. For every ϕ, $\varphi \in V$, let $D = \{t \in \mathbf{C} : \phi(t) = \varphi + t\phi \in V\}$ and $\mu(t) = \mu_{\varphi(t)}$. From Ahlfors-Weil theorem, we have $\mathbf{B}^{-1}(\varphi(t)) = [w_{\mu(t)}]$. The Beltrami coefficient of $w^{\mu(t)} \circ \omega^{-1}$ is given by

$$\lambda(t) = \frac{\overline{\omega_z}}{\omega_z} \frac{\mu_t - \mu_\omega}{1 - \overline{\mu_\omega}\, \mu_t} \circ \omega^{-1} \qquad \text{on H.}$$

Then $F(\varphi(t)) = \{w_{\lambda(t)}, z\}$. Since l(t) is holomorphic with respect to t, $F(\varphi(t))$ is biholomorphic on D by theorem 5.11. So F is holomorphic on V. By using the lemma 5.12 and inverse mapping theorem, F is biholomorphic on V.

□

3. Teichmüller Modular Groups

There are two ways to introduce the Teichmüller modular groups.

Definition. The *Teichmüller modular group* Mod(R) of R is the quotient group of the group of all qc self-mappings f_0 of R over the normal subgroup of f_0 homotopic to id.

The action $[f_0]_*$ of $[f_0] \in \mathrm{Mod}(R)$ on T(R) is given by

$$[f_0]_*([S,f]) = [S, f \circ f_0^{-1}] \qquad \forall [S,f] \in T(R)$$

which is called a *Teichmüller modular transformation* of T(R).

Let Γ be a Fuchsian model of R. Let w_i be a lift of a qc self-mapping f_i of R with $\omega_i \Gamma \omega_i^{-1} = \Gamma$ for i = 1, 2. By the same argument as in the proof of lemma 4.1, we can check that $[f_1] = [f_2]$ in Mod(R) $\Leftrightarrow$ $\omega_2 = \omega_1 \circ \gamma_0$ on $\mathbf{R}$ for some $\gamma_0 \in \Gamma$.

Definition. ω_1 and ω_2 satisfying $\omega_i \Gamma \omega_i^{-1} = \Gamma$ for i = 1, 2 are said to be *equivalent* if there exists $\gamma_0 \in \Gamma$ such that $\omega_2 = \omega_1 \circ \gamma_0$ on **R**. The *Teichmüller modular group* Mod(Γ) of Γ is the group of all equivalence classes [ω]. The action $[\omega]_*$ of $[\omega] \in$ Mod(Γ) on T(Γ) is give by

$$[\omega]_*([w^\mu]) = [\alpha \circ w^\mu \circ \omega^{-1}] \qquad \forall\ [w^\mu] \in T(\Gamma),$$

where $\alpha \in$ Aut(H) such that $\alpha \circ w^\mu \circ \omega^{-1}$ fixes each 0, 1, and ∞.

Moreover, $[\omega]_* \in$ Aut(T(Γ)) induces a biholomorphic automorphism $<\omega>_*$ of $T_\beta(\Gamma)$ defined by

$$<\omega>_* ([w_\mu]) = [w_\nu] \qquad \forall [w_\mu] \in T_\beta(\Gamma)$$

where ν is the Beltrami coefficient of $\alpha \circ w^\mu \circ \omega^{-1}$.

Remark. Mod(R) ≅ Mod(Γ).

Since T(R) ≅ T(Γ), the Teichmüller-distance on T(R) induces the Teichmüller-distance on T(Γ). So we have

Theorem 5.13. *For every* $[\omega] \in Mod(\Gamma)$, $[\omega]_*$ *is an isometry with respect to the Teichmüller-distance.*

Moduli Sets

We first observe that

Proposition 5.15. *The set of hyperbolic lengths of all closed geodesics on a Riemann surface of genus g ≥ 2 is discrete in* ***R*** *with hyperbolic length l.*

This proposition follows from the following fact : For every compact set K in H and M > 0, there exists finitely many $\gamma \in \Gamma$ such that $\min_{z \in K} \rho(z,\gamma(z)) \leq M$.

By the proposition 5.15, we get

Corollary. $\left\{ tr^2(\gamma) = 4\cosh^2\left(\frac{l(L_\gamma)}{2}\right) : \gamma \in \Gamma \right\}$ *is discrete in* $\boldsymbol{R}$.

From Theorem 2.9, a set of hyperbolic lengths of closed geodesics on R induces a normalized Fuchsian model Γ of R. Now, we state a weaker result as follows.

Proposition 5.16. *Let* Γ *be a Fuchsian model of a Riemann surface of genus* $g \geq 2$. *Let* $\{\gamma_j\}_{j=1}^{m}$ *be a system of generators of* Γ *such that*

1. γ_1 *has the repelling fixed point 0 and the attractive fixed point* ∞, *and*

1. γ_2 *has the repelling fixed point* $r < 0$ *and the attractive fixed point 1.*

Then each γ_j *is determined by the absolute values of trace of elements in*

$$\mathcal{G} = \{ \gamma_1 \circ \gamma_2,\ \gamma_1^{\pm 1} \circ \gamma_k,\ \gamma_2^{\pm 1} \circ \gamma_k,\ (\gamma_1 \circ \gamma_2)^{\pm 1} \circ \gamma_k \}$$

where $j = 1,\ldots m$ *and* $k = 3,\ldots m$.

Discontinuity Teichmüller Modular Groups

Theorem 5.17. *Mod(Γ) acts properly discontinuously on T(Γ) as a subgroup of the biholomorphic automorphism group Aut(T(Γ)).*

Proof. Suppose that Γ has a system of generators on Proposition 5.16. If not. There exists a sequence of $\{f_n\}_{j=1}^{\infty}$ of mutually distinct elements in Mod(Γ) such that, $p_n \in T(\Gamma) \rightarrow p_0 \in T(\Gamma)$ as $n \rightarrow \infty$, $f_n(p_n) \rightarrow q_0 \in T(\Gamma)$ as $n \rightarrow \infty$.

For each n, set $h_n = f_{n+1}^{-1} \circ f_n$. Since $T(\Gamma) \cong T_B(\Gamma)$, $\{f_n\}_{j=1}^{\infty}$ is a normal family. We may assume that $\{f_n\}_{j=1}^{\infty}$ converges uniformly on compact sets in T(Γ) to a holomorphic mapping f_0. So $f_0(p_0) = q_0$. Thus, we can see that $h_n(p_0) \rightarrow p_0$ as $n \rightarrow \infty$.

For each n, $h_n = [\omega_n]_*$ with some qc self-mapping ω_n of H such that $\omega_n \Gamma \omega_n^{-1} = \Gamma$. Then we have

$$[\omega_n]_* \,([\mathrm{id}]) = [\, \alpha_n \circ \omega_n^{-1} \,],$$

where $\alpha_n \in \mathrm{Aut}(H)$ such that $\alpha_n \circ \omega_n^{-1}$ fixes each 0, 1, and ∞. So $\{(\alpha_n \circ \omega_n^{-1}) \circ \gamma \circ (\alpha_n \circ \omega_n^{-1})^{-1} \to \gamma$ as $n \to \infty$. We have

$$\lim_{n\to\infty} \mathrm{tr}^2(\omega_n^{-1} \circ \gamma \circ \omega_n) = \mathrm{tr}^2(\gamma), \qquad \gamma \in \Gamma.$$

Since $\{ \mathrm{tr}^2(\gamma) : \gamma \in \Gamma \}$ is discrete on $\mathbf{R}$, and every $\omega_n \circ \gamma \circ \omega_n^{-1} \in \Gamma$, we have

$$\lim_{n\to\infty} \mathrm{tr}^2(\omega_n^{-1} \circ \gamma \circ \omega_n) = \mathrm{tr}^2(\gamma), \qquad \gamma \in \mathcal{G} \text{ for large n.}$$

Here, for large n there exists $\beta_n \in \mathrm{Aut}(H)$ such that

$$\omega_n^{-1} \circ \gamma \circ \omega_n = \beta_n^{-1} \circ \gamma \circ \beta_n, \quad \gamma \in \Gamma.$$

This shows that $\beta_n \in N(\Gamma)$ the normalizer of Γ in Aut(H), and $[\omega_n]_* = [\beta_n]_*$. Thus every $[\omega_n]_*$ fixes [id] of $T(\Gamma)$.

By definition, we have

isotropy subgroup of $\mathrm{Mod}(\Gamma)$ at [id] = $N(\Gamma)/\Gamma = \mathrm{Aut}(H/\Gamma)$.

H. A. Schwarz proved that $\mathrm{Aut}(H/\Gamma)$ is a finite group. [Fr,Ka] So $\{[\omega_n]_*\}_{n=1}^{\infty}$ is a finite set. This contradicts that $\{f_n\}_{n=1}^{\infty}$ consists of infinite elements.

□

Theorem 5.18. *The moduli space M_g of closed Riemann surface of genus $g \geq 2$ has a normal complex analytic space structure of dimension 3g-3.* [Ba]

4. Classification of Teichmüller Modular Transformations

In this section, we let R be an oriented closed differentiable surface of genus $g \geq 2$. Fix a Riemann surface R^* whose underlying surface is R.

Extermal Problems

Bers' extermal problem for Teichmüller modular transformations

For every Teichmüller modular transformation χ of $T(R^*)$, we set

$$a(\chi) = \inf_{p \in T(R^*)} d(p, \chi(p)).$$

Then we want to find $p_\chi \in T(R^*)$ which is called a *χ-minimal point* such that

$$a(\chi) = d(p_\chi, \chi(p_\chi)).$$

Teichmüller modular transformations χ is classified into 4 types:

1. χ is elliptic if $a(\chi) = 0$ and there exists a χ-minimal point.
2. χ is parabolic if $a(\chi) = 0$ but there exists no χ-minimal point.
3. χ is hyperbolic if $a(\chi) > 0$ and there exists a χ-minimal point.
4. χ is pseudo-hyperbolic if $a(\chi) > 0$ but there exists no χ-minimal point.

This classification is independent of the choice of the complex structure on R which is used to define the Teichmüller space $T(R^*)$.

Bers' extermal problem for complex structures

For every complex structure σ on R and $f : R \to R$. The maximal dilatation of f $K_\sigma(f) = \infty$ if f is not a qc mapping. Then we want to find a complex structure σ_0 on R and $f_0 : R \to R$ homotopic to f such that

$$K_{\sigma_0}(f_0) \leq K_{\sigma_1}(f_1)$$

for every complex structure σ_1 on R and $f_1 : R \to R$ homotopic to f.

If there exists a solution, σ_0 is called an *f-minimal complex structure* and

$f_0 : R_{\sigma_0} \to R_{\sigma_0}$ is called an *absolutely extermal self-mapping* of R_{σ_0}.

The following proposition relates the two extermal problems.

Proposition 5.19. *For $f : R \to R$, a complex structure σ is f-minimal if and only if $[R_\sigma, id] \in T(R^*)$ corresponding to σ is $[f]_*$-minimal.*

Elliptic and Hyperbolic Transformations

Applying the above proposition, we have the following theorem.

Theorem 5.20. *Let $f : R \to R$. Then there exists an f-minimal complex structure $\Leftrightarrow$ $[f]_*$ corresponding to f is elliptic or hyperbolic.*

Proof. see [Wo 4].

Theorem 5.21. *χ is elliptic $\Leftrightarrow$ χ is periodic.*

Definition. A set $\{C_j\}_{j=1}^{n}$ of mutually disjoint simple closed curves on R is said to be *admissible* if every C_j is freely homotopic to none of $\{C_k, C_k^{-1}\}_{k\neq j}$, and is not null homotopic.

A self-mapping f of R is said to be *reduced* by $\{C_j\}_{j=1}^{n}$ if $\{C_j\}_{j=1}^{n}$ is admissible and $f(C_1\cup\ldots C_n) = C_1\cup\ldots C_n$.

A self-mapping f of R is said to be *reducible* if f is homotopic to a reduced mapping and *irreducible* if not.

We are going to prove the following theorem.

Theorem 5.22. *If f is an irreducible self-mapping of R, then $[f]_*$ is either elliptic or hyperbolic.*

Before proving this theorem, several lemmas are prepared as follows.

Lemma 5.23 (Wolpert) *Let f be a qc mapping of a Riemann surface S_1*

onto S_2, and C be a simple closed geodesic on S_1 with hyperbolic length l_1.

Then f(C) is freely homotopic to a closed geodesic with hyperbolic length l_2 such that

$$l_2 \leq K(f)\, l_1$$

where K(f) is the maximal dilatation of f. [Wo 2]

Lemma 5.24. *There exists a constant $\delta_0(\gamma) > 0$ such that any 2 simple closed geodesics on a Riemann surface of genus g are disjoint provided that hyperbolic length of them are less than δ_0.* [Mat]

Lemma 5.25. Let C be a simple closed geodeisc with hyperbolic length l on a Riemann surface S of genus g ≥ 2. Then every irreducible mapping f : S → S satisfies

$$K(f) \geq \sqrt[3g-3]{\frac{\delta_0}{l}}.$$

By using these lemmas, we can have the following lemma.

Lemma 5.26. (Mumford's compactness theorem) *Let $\{p_j\}_{j=1}^{\infty}$ be a sequence in T(R*), and S_j be the Riemann surface corresponding to p_j for every j. Suppose that there exists $\delta > 0$ such that the hyperbolic length of any simple closed geodesic on each S_j is greater than δ.*

Then there exists a subsequence $\{p_{j_n}\}_{n=1}^{\infty}$ and a sequence $\{\chi_n\}_{n=1}^{\infty}$ of Teichmüller modular transformation of T(R) such that $\chi_n(p_{j_n})$ converges in T(R*) as $n \to \infty$.* [Be]

Now, we use the above lemmas to prove the theorem 5.20.

Proof of theorem 5.20. Take a sequence $\{p_j\}_{j=1}^{\infty}$ in T(R*) such that

$$d(p_j,[f]_*(p_j)) \to a([f]_*) \qquad \text{as} \qquad j \to \infty.$$

For every j, let σ_j be the complex structure corresponding to p_j and $[f]_*(p_j)$, and $h_j : R_{\sigma_j} \to R_{\sigma_j}$ be the Teichmüller mapping homotopic to f. Then we have

$$\log K(h_j) \to a([f]_*) \quad \text{as} \quad j \to \infty.$$

In particular, there exists a constant A such that $K(h_j) < A$ for all j. By lemma 5.25, the hyperbolic length of any simple closed geodesic on each R_{σ_j} is greater than $\delta_0 A^{3-3g}$. By lemma 5.26, we assume that there exists a sequence $\{\chi_n\}_{n=1}^{\infty}$ of Teichmüller modular transformation of T(R*) such that

$$q_j = \chi_n(p_j) \to q \in T(R^*) \quad \text{as} \quad j \to \infty.$$

Since each χ_j isometric with respect to d, we have

$$d(q_j, \chi_j[f]_* \chi_j^{-1}(p_j)) \to a([f]_*) \quad \text{as} \quad j \to \infty.$$

We assume $\chi_j[f]_*\chi_j^{-1}(p_j)$ converges to $q' \in T(R^*)$ as $j \to \infty$, for T(R*) is complete finite-dimensional with respect to d. Since each $\chi_j[f]_*\chi_j^{-1}$ is isometric, we have

$$\chi_j[f]_*\chi_j^{-1}(q) \to q' \quad \text{as} \quad j \to \infty.$$

Hence, we assume that $\chi_j[f]_*\chi_j^{-1}$ has the same action on T(R*) for $j \geq k$ for some k. Then

$$d(q, \chi_k[f]_*\chi_k^{-1}(q)) = d(\chi_j^{-1}(q)[f]_*\chi_j^{-1}(q)) = a([f]_*).$$

Therefore, there exists an $[f]_*$-minimal point and then f is elliptic or hyperbolic.

□

Absolutely Extermal Mappings

T(R*) is a *straight line space* in the sense of Buseman. [Mas] This means $p_1, p_2 \in T(R^*)$ lie on one *straight line* L which is an isometric image of **R** into T(R*) equipped with respect to d, and contains all point p such that

$$d(p_1,p) + d(p,p_2) = d(p_1,p_2).$$

Firstly, we have the useful theorem as follows.

Theorem 5.27. *If χ is of infinite order, then $p \in T(R^*)$ is χ-minimal $\Leftrightarrow$ χ leaves a straight through p invariant.*

Proof. $\Rightarrow$ Since χ is of infinite order, p, $\chi(p)$ and $\chi^2(p)$ are distinct. Let p_1 and p_2 be the midpoint of p & $\chi(p)$ and $\chi(p)$ & $\chi^2(p)$ respectively. We have

$$d(p_1,p_2) \le d(p_1,\chi(p)) + d(\chi(p),p_2) = a(\chi).$$

On the other hand, $d(p_1,p_2) \ge a(\chi)$ since $p_2 = \chi(p_1)$. So $\chi(p)$ is the midpoint of p_1 and p_2. Thus the straight line on which p, p_1, $\chi(p)$, p_2 and $\chi^2(p)$ lie is invariant under χ.

$\Leftarrow$ For every $p' \in T(R^*)$ and $n \in \mathbf{N}$, we have

$$n\, d(p,\chi(p)) = d(p_1,\chi^n(p)) \le 2\, d(p,p') + n\, d(p',\chi(p')).$$

Since n is arbitrary, we have $d(p,\chi(p)) \le d(p',\chi(p'))$. Thus p is a χ-minimal point.

□

Corollary. *For a non-periodic $[f] \in Mod(R^*)$, $[f]_*$ is hyperbolic $\Leftrightarrow$ $[f]_*$ leaves a straight line invariant.*

Theorem 5.28. *Let S be a Riemann surface of genus $g \ge 2$. Then $f : S \to S$ is absolutely extermal $\Leftrightarrow$ f is either conformal or Teichmüller such that f^2 is a Teichmüller mapping with $K(f^2) = K(f)^2$.*

Proof. Assume f is Teichmüller . Let σ be the complex structure of S. By theorem 5.19, we have

f is absolutely extermal.

$\Leftrightarrow$ $[\sigma] \in T(R^*)$ $[f]_*$-minimal.

$\Leftrightarrow$ $d([\sigma],[f]_*([\sigma])) + d([f]_*([\sigma]),[f]_*{}^2([\sigma])) = d([\sigma]),[f]_*{}^2([\sigma]))$

$\Leftrightarrow$ $K(f)^2 = K(h)$ where h is Teichmüller and is homotopic to f^2.

$\Rightarrow$ Since $K(h) \leq K(f^2) \leq K(f)^2$, f^2 is Teichmüller with $K(f^2) = K(f)^2$.

$\Leftarrow$ We have

$$d([\sigma],[f]_*([\sigma])) + d([f]_*([\sigma]),[f]_*{}^2([\sigma])) = d([\sigma]),[f]_*{}^2([\sigma])).$$

Hence, $d([\sigma],[f]_*([\sigma])) \leq d([\sigma]),[f]_*{}^2([\sigma]))$. So f is absolutely extermal.

□

For reducible mapping, we have the following theorem.

Theorem 5.29. *Let $f : R \to R$ be reducible. If f is not homotopic to a periodic mapping,, then $[f]_*$ is either parabolic or pseudo-hyperbolic.*

Based on the above results, we get a necessary and sufficient conditions for the existence of f-minimal complex structure.

Theorem 5.30. *Let $f : R \to R$. An f-minimal complex structure exists $\Leftrightarrow$ f is either homotopic to a periodic mapping or irreducible.*

CHAPTER 6

Weil-Petersson Metric

1. Petersson Scalar Product and Reproducing formula

All results can be formulated in H and Δ. We shall use either formulation interchangeably.

Definition. A Hermitian scalar product on $A_2(H,\Gamma)$ is defined by

$$<\varphi,\psi>_R = \iint_F (\text{Im } z)^2 \varphi(z) \overline{\psi(z)} \, dxdy, \qquad \forall \varphi, \psi \in A_2(H,\Gamma)$$

where F is a fundamental domain for the Fuchsian model Γ in H. $< , >_R$ is called the *Petersson scalar product* on $A_2(H,\Gamma)$.

Since the Riemann surface R is compact, $A_2(H,\Gamma)$ is a Hilbert space with the scalar product. Let $A_2(H)$ denote the Hilbert space of holomorphic functions ψ on H such that

$$\|\psi\|^2 = \iint_F (\text{Im } z)^2 |\psi(z)|^2 dxdy < \infty.$$

Definition. The *reproducing kernel* for $A_2(H)$ is defined by

$$K_H(z,\zeta) = \frac{12}{\pi(\overline{z} - \zeta)^4}, \qquad z, \zeta \in H.$$

We use the kernel to obtain the reproducing formula. In the following, we write $\zeta = \xi + i\eta$.

Theorem 6.1. (Reproducing formula) *Every $\varphi \in A_2(H,\Gamma)$ satisfies*

$$\varphi(z) = \iint_H (Im\ z)^2\ \varphi(z)\ \overline{K_H(z,\zeta)}\ d\xi d\eta, \quad z \in H.$$

Proof. It is sufficient to show that for every $\psi \in A_2(\Delta,\Gamma')$

$$\psi(z) = \iint_\Delta \lambda(\zeta)^{-2}\ \psi(\zeta)\ \overline{K_\Delta(z,\zeta)}\ d\xi d\eta, \quad z \in \Delta \quad (*)$$

where $K_\Delta(z,\zeta) = \dfrac{12}{\pi(1 - \bar{z}\zeta)^4}$ is the *reproducing kernel* for $A_2(\Delta)$, $\lambda(z) = \dfrac{2}{1 - |z|^2}$

and $\Gamma' = T\Gamma T^{-1}$ with $T = \dfrac{z - i}{z + i}$.

Since H/Γ is compact, $|\psi|\lambda^{-2}$ is bounded on Δ. So (*) converges absolutely for any z.

By the mean value theorem for a holomorphic function, we have

$$\psi(0) = \iint_\Delta \lambda(\zeta)^2\, \psi(\zeta)\, \overline{K_\Delta(0,\zeta)}\, d\xi d\eta, \quad (**).$$

So (*) is true for z = 0.

For every $z \in \Delta$, choose $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(0) = z$. If we set $\psi_0(0) = (\psi \circ \gamma)\, \gamma'^2$, then $\psi_0 \in A_2(\Delta, \gamma^{-1}\Gamma'\gamma)$, and satisfies $\psi_0(0) = \psi(z)\, \gamma'(0)^2$. Applying (**) to ψ_0, we have

$$\psi_0(0) = \iint_\Delta \lambda(\gamma^{-1}(w))^2\, \psi_0(\gamma^{-1}(w))\, \overline{K_\Delta(\gamma^{-1}(z), \gamma^{-1}(w))}\, |\gamma^{-1}(w)|^2 du dv.$$

Then we get

$$\psi(z) = \frac{\psi_0(0)}{\gamma'(0)^2} = \iint_\Delta \lambda(\zeta)^{-2}\, \psi(\zeta)\, \overline{K_\Delta(z,\zeta)}\, d\xi d\eta, \quad z \in \Delta$$

□

Poincarè Series

We may rewrite (*) as

$$\psi(z) = \sum_{\gamma \in \Gamma'} \left[\iint_F \lambda(\zeta)^{-2}\, \psi(\zeta)\, \overline{K_\Delta(\gamma(z),\zeta)}\, d\xi d\eta \right] \gamma'(z)^2.$$

where F is a relative compact fundamental domain for Γ' in Δ.

Definition. For a holomorphic function f on Δ, we define the *Poincarè series* of f for Γ' by

$$\theta f(z) = \sum_{\gamma \in \Gamma'} f(\gamma(z))\, \gamma'(z)^2, \qquad z \in \Delta.$$

Theorem 6.2. *Let f be an integrable holomorphic function on Δ and let θf be the Poincaré series of f for a Fuchsian group Γ' on Δ. Then θf converges absolutely and uniformly on compact sets in Δ, and belongs to $A_2(\Delta,\Gamma')$.*

Proof. For any compact set K in Δ and $z \in K$,

$$\sum_{\gamma\in\Gamma'} |f(\gamma(z))\ \gamma'(z)^2| \leq \frac{1}{\pi r^2}\iint_\Delta |f(z)|d\xi d\eta,$$

hence absolutely and uniformly convergence follow. One also check easily $\theta f(\gamma(z))\ \gamma'(z)^2 = \theta f(z)$. □

Corollary 1. *For every $\psi \in A_2(\Delta,\Gamma')$, there exists a holomorphic function f on a neighborhood of $\bar{\Delta}$ such that $\varphi = \theta f$ on Δ.*

Corollary 2. *Every $\psi \in A_2(H,\Gamma)$ is written as*

$$\varphi(z) = \sum_{\gamma\in\Gamma}\left[\iint_F (Im\zeta)^2\ \varphi(\zeta)\ \overline{K_H(\gamma(z),\zeta)}\ d\xi d\eta\right] \gamma'(z)^2,\ z \in H,$$

and the series converges absolutely and uniformly on compact sets in H, where F is a fundamental domain for Γ in H.

Bergman Projection

By using the reproducing kernel of $A_2(H)$, we shall construct holomorphic quadratic differentials on the Riemann surface H/Γ.

Definition. A function f is called a *measurable automorphic form* with respect to Γ on H if f is measurable on H, and satisfies

$$f(\gamma(z))\ \gamma'(z)^2 = f(z), \qquad z \in H, \gamma \in \Gamma.$$

Let $L_2^\infty(H,\Gamma)$ be the set of all measurable automorphic form with respect to Γ on H with

$$\|f\|_\infty = \operatorname*{ess\,sup}_{z\in H} (\mathrm{Im}\ z)^2|f(z)| < \infty.$$

It can be checked that $L_2^\infty(H,\Gamma)$ is a Banach space with this norm and $A_2(H,\Gamma)$ is a closed subspace of $L_2^\infty(H,\Gamma)$. The Petersson scalar product on $A_2(H,\Gamma)$ is extended to $L_2^\infty(H,\Gamma)$. Then we have the theorem.

Theorem 6.3. *For any $f \in L_2^\infty(H,\Gamma)$, set*

$$(\beta_2 f)(z) = \iint_H (\mathrm{Im}\, z)^2 f(z)\, \overline{K_H(z,\zeta)}\, d\xi d\eta, \qquad z \in H \quad (*).$$

Then $\beta_2 f \in A_2(H,\Gamma)$, and satisfies $\beta_2 f = \theta g$ on H, where

$$g(z) = \iint_F (\mathrm{Im}\, z)^2 f(z)\, \overline{K_H(z,\zeta)}\, d\xi d\eta, \text{ and}$$

F is a fundamental domain for Γ in H.

Proof. Since $\|f\|_\infty < \infty$ and $K_H(z,\zeta) = \dfrac{12}{\pi(\bar{z} - \zeta)^4}$, (*) converges absolutely. $\beta_2 f$ is holomorphic on H and the belongs to $A_2(H,\Gamma)$. By using corollary 2, we get the assertion.

□

The bounded linear operator $\beta_2 : L_2^\infty(H,\Gamma) \to A_2(H,\Gamma)$ is called the *Bergman projection.*

Theorem 6.4. *Any $f, g \in L_2^\infty(H,\Gamma)$ satisfy*

$$<\beta_2 f, g>_R = <f, \beta_2 g>_R.$$

Remark. $\quad \beta_2(\lambda_H^2 \bar{\mu})(z) = -2\, \overline{\dot{\phi}_0[\mu](\bar{z})}\,, \quad z \in H.$

2. Infinitesimal Theory of Teichmüller Spaces

The Tangent Space at the Base Point

Definition. The holomorphic tangent space of $T(\Gamma)$ at the base point is denoted by $T_0(T(\Gamma))$.

Since $\phi_0 : B(H,\Gamma) \to T_0(T_B(\Gamma)) = T_0(T(\Gamma))$ is a surjective mapping, we have

$$T_0(T(\Gamma)) \cong B(H,\Gamma)/\, N(\Gamma)$$

where $N(\Gamma) = \mathrm{Ker}\phi_0$.

Definition. A linear functional defined by

$$\Lambda_\mu(\varphi) = (\mu,\varphi)_{H/\Gamma} = \iint_F \mu(z)\, \varphi(z)\, dxdy, \qquad \varphi \in A_2(H,\Gamma)$$

where F is a fundamental domain for Γ in H.

Lemma 6.5 (Teichmüller). $\mu \in N(\Gamma) \Leftrightarrow \Lambda_\mu = 0.$

The Teichmüller lemma can be verified by direct computation.

Theorem 6.6. *The mapping* $\Lambda : B(H,\Gamma) \to A_2(H,\Gamma)^*$ *given by* $\Lambda(\mu) = \Lambda_\mu$ *induces an isomorphism of* $B(H,\Gamma)/\, N(\Gamma)$ *onto* $A_2(H,\Gamma)^*$. *In particular,*

$$T_0(T(\Gamma)) \cong A_2(H,\Gamma)^*.$$

Proof. Teichmüller lemma asserts that $\mathrm{Ker}\Lambda = N(\Gamma)$. Every $f \in A_2(H,\Gamma)^*$ is written as $<\ ,\ \psi>_{H/\Gamma}$ for some $\psi \in A_2(H,\Gamma)$. Putting $\mu = (\mathrm{Im} z)^2\, \bar{\psi}$, we have $\mu \in B(H,\Gamma)$ and $\Lambda_\mu = f$. So Λ is surjective. Λ induces an isomorphism of $B(H,\Gamma)/\, N(\Gamma)$ onto $A_2(H,\Gamma)^*$.

$\square$

Harmonic Beltrami Differentials

Definitions. For every $\varphi \in A_2(H,\Gamma)$, the *harmonic Beltrami differential* in $B(H,\Gamma)$ induced by φ is defined as

$$\mu[\varphi](z) = (\mathrm{Im} z)^2\, \overline{\varphi(z)}, \qquad z \in H.$$

For every $\mu \in B(H,\Gamma)$, $\varphi[\mu] \in A_2(H,\Gamma)$ is defined by

$$\varphi[\mu](z) = -2\ \overline{\dot{\phi}_0[\mu](\overline{z})} = \beta(\frac{\overline{\mu}}{(\mathrm{Im}\ z)^2})(z), \qquad z \in H.$$

By the reproducing formula, we have

$$\varphi[\mu[\varphi]] = \varphi, \quad \varphi \in A_2(H,\Gamma).$$

For any $\mu \in B(H,\Gamma)$, the *harmonic Beltrami differential* induced by μ is defined as

$$H[\mu](z) = (\mathrm{Im} z)^2\ \overline{\varphi[\mu](z)}\ .$$

Let $HB(H,\Gamma)$ be the vector space of all harmonic Beltrami differentials. We have a surjective mapping $H : B(H,\Gamma) \to HB(H,\Gamma)$ which is given by

$$H[\mu](z) = (\mathrm{Im} z)^2\ \beta(\frac{\overline{\mu}}{(\mathrm{Im}\ z)^2}), \quad \mu \in B(H,\Gamma).$$

Then the following results can be easily verified:

1. $\mathrm{Ker} H = N(\Gamma)$,
2. $H[\mu[\varphi]] = \mu[\varphi], \qquad \varphi \in A_2(H,\Gamma)$.
3. $\varphi[H[\mu]] = \varphi[\mu], \qquad \mu \in B(H,\Gamma)$.
4. $H^2 = H$.

Based on the above results, we have

Theorem 6.7. *$B(H,\Gamma) = HB(H,\Gamma) \oplus N(\Gamma)$.*

$\dot{\phi}_0$ at the base point induces the isomorphism $\dot{\phi}_0 : HB(H,\Gamma) \to T_0(T_B(\Gamma))$. In particular, $T_0(T(\Gamma)) \cong HB(H,\Gamma)$.

Moreover, $(\mu,\varphi)_R = (H[\mu],\varphi)_R, \qquad \varphi \in A_2(H,\Gamma),\ \mu \in B(H,\Gamma)$.

Proof. Take $\mu \in N(\Gamma) = \mathrm{Ker} H$. We have $H[\mu] = 0$.

If $\mu \in HB(H,\Gamma)$, then there exists $\nu \in B(H,\Gamma)$ such that $\mu = H[\nu] = H^2[\nu] = 0$. So

$$HB(H,\Gamma) \cap N(\Gamma) = \{0\}.$$

Every $\mu \in B(H,\Gamma)$ is decomposed into

$$\mu = H[\mu] + (\mu - H[\mu]).$$

Since $H^2 = H$, $\mu - H[\mu] \in \mathrm{Ker}\, H = N(\Gamma)$. So $B(H,\Gamma) = HB(H,\Gamma) \oplus N(\Gamma)$.

$\square$

Tangent Space of T(Γ)

Definition. The translation mapping $[w^\nu]_* : T(\Gamma) \to T(\Gamma^\nu)$ induces an isomorphism of $T_p(T(\Gamma)) \to T_0(T(\Gamma^\nu))$ where $p = [w^\nu]$ and $\Gamma^\nu = w^\nu\, \Gamma\, (w^\nu)^{-1}$.

The description of the isomorphism is given as follows.

Define

$$\kappa = F(\lambda) = \left(\frac{w^\nu{}_z \quad \lambda - \nu}{\overline{w^\nu{}_z} \quad 1 - \bar{\nu}\lambda} \right) \circ \quad (w^\nu)^{-1},$$

we have $\qquad [w^\nu]_*([w^\lambda]) = [w^\kappa], \qquad [w^\lambda] \in T(\Gamma).$

Let $L^\nu : B(H,\Gamma) \to B(H,\Gamma^\nu)$ be an isomorphism given by

$$L^\nu[\mu] = \lim_{t \to 0} \frac{F(\nu + t\mu) - F(\nu)}{t}.$$

Also, let $H^\nu : B(H,\Gamma^\nu) \to HB(H,\Gamma^\nu)$ be the projection. Then we have the following proposition.

Proposition 6.8. *For every $p = [w^\nu] \in T(\Gamma)$, $H^\nu \circ L^\nu : B(H,\Gamma)/Ker\dot{\phi}_\nu \to HB(H,\Gamma)$ is an isomorphism and is independent of the choice of a representative w^ν of p.*

3. Weil-Petersson Metric

Definition. The inner product of $\mu_1, \mu_2 \in B(H,\Gamma)$ is defined by

$$h(\mu_1,\mu_2) = \iint_F \frac{\mu_1(z)\,\overline{\mu_2(z)}}{(\mathrm{Im} z)^2}\,dxdy,$$

where F is a relatively compact fundamental domain for Γ in H. Then we have

$$h(H[\mu_1],H[\mu_2]) = <\varphi[\mu_2],\varphi[\mu_1]>_R.$$

Lemma 6.9. *For $\mu_1,\mu_2 \in B(H,\Gamma)$, the following hold:*

$$h(H[\mu_1],H[\mu_2]) = h(H[\mu_1],\mu_2) = h(\mu_1,H[\mu_2]),$$

$$h(H[\mu_1],H[\mu_2]) = (H[\mu_1],\varphi[\mu_2])_R = (\mu_1,\varphi[\mu_2])_R.$$

A Hermitian scalar product on $T_p(T(\Gamma))$ is induced by h^ν on $B(H,\Gamma^\nu)$ The inner product of $H^\nu \circ L^\nu[\mu_1]$ and $H^\nu \circ L^\nu[\mu_2]$ in $HB(H,\Gamma^\nu) \cong T_p(T(\Gamma))$ is given by

$$h^\nu(H^\nu \circ L^\nu[\mu_1]\,,\, H^\nu \circ L^\nu[\mu_2]), \qquad \mu_1,\mu_2 \in B(H,\Gamma).$$

We study the dependence of the inner product with respect to p in a neighborhood of 0. Take a basis $\{\varphi_j\}_{j=1}^{3g-3}$ for $A_2(H,\Gamma)$, and let

$$\nu_j = \mu[\varphi_j], \qquad j = 1, \ldots 3g\text{-}3.$$

Then $\{\nu_j\}_{j=1}^{3g-3}$ is a basis for $HB(H,\Gamma)$. Put

$$\nu(t) = \sum_{i=1}^{3g-3} t^j\,\nu_j, \qquad t(t^1,\ldots t^{3g-3}) \in D$$

where D is an open neighborhood of 0 in $\mathbf{C}^{3g-3}$.

For a sufficiently small D, we assume the mapping $\phi_0 : D \to T_B(\Gamma)$ defined by $\phi_0(t) = \phi(\nu(t))$ is biholomorphic onto an open neighborhood of the base point in $T_B(\Gamma)$. Let

$$\nu_j(t) = L^{\nu(t)}[\nu_j], \qquad t \in D.$$

Then $\{H^{v(t)}[v_j]\}_{j=1}^{3g-3}$ is a basis of $HB(H,\Gamma^{v(t)})$ for every $t \in D$. Thus the inner product $h_{j\bar{k}}(t)$ of tangent vectors $\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k} \in T_{p(t)}(T((\Gamma))$ with $p(t) = [w^{v(t)}]$ is given by

$$h_{j\bar{k}}(t) = h^v(t)[H^{v(t)}[v_j(t)], H^{v(t)}[v_k(t)]) = (v_j(t), \varphi[v_k(t)])_{H/\Gamma^{v(t)}}.$$

Then we have the following theorem.

Theorem 6.10. *Under the above circumstances, each $h_{j\bar{k}}(t)$ is of class C^∞ on D.*

Definition. The inner product on the tangent bundle of $T(\Gamma)$ is called the *Weil-Petersson metric* h_{WP} on $T(\Gamma)$. Locally, h_{WP} is written as

$$ds_{WP}^2 = 2 \sum_{j,k=1}^{3g-3} h_{j\bar{k}}(t) dt^j \overline{dt^k}.$$

By the definition, it can be shown that h_{WP} on $T(\Gamma)$ is invariant under the action of $Mod(\Gamma)$.

Kählerity of the Weil-Petersson Metric

Definition. Let g_{WP} be the Riemannian metric on $T(\Gamma)$ induced by h_{WP}. Then

$$g_{WP}(X,Y) = 2 \operatorname{Re} h_{WP}(X,Y),$$

where X and Y on the left hand side are real tangent vectors, and those on the right hand side are holomorphic tangent vectors under identifying the real tangent space of $T(\Gamma)$ at p and $T_p(T(\Gamma))$. g_{WP} is called the *Weil-Petersson Riemannian metric* on $T(\Gamma)$.

The Weil-Petersson form ω_{WP} of h_{WP} is defined by

$$\omega_{WP}(X,Y) = g_{WP}(iX,Y),$$

$$= -2\,\mathrm{Im}\, h_{WP}(X,Y) \quad X, Y \in T_p(T(\Gamma)).$$

Locally, ω_{WP} is positive (1,1)-form and is represented by

$$\omega_{WP} = i \sum_{j,k=1}^{3g-3} h_{j\bar{k}}(t)dt^j \wedge \overline{dt^k}.$$

Definition. h_{WP} is a *Kähler metric* if $d\omega_{WP} = 0$, which is equivalent to

$$\frac{\partial h_{j\bar{k}}}{\partial t^l} = \frac{\partial h_{l\bar{k}}}{\partial t^j} \text{ on } D$$

where $j, k, l = 1, \ldots 3g-3$.

Theorem 6.11. (Ahlfors) *The Weil-Petersson metric is Kählerian.*

Sketch of proof. It is sufficient to prove that $\frac{\partial h_{j\bar{k}}}{\partial t^l} = \frac{\partial h_{l\bar{k}}}{\partial t^j}$ at the base point.

Let $f^t = w^{\nu(t)}$, $\Gamma^{\nu(t)} = f^t\Gamma f^{t-1}$, $R(t) = H/\Gamma^{\nu(t)}$, and $F(t) = f^t(F)$.

Set

$$K(z,\zeta) = \frac{1}{(z-\zeta)^2} \qquad \text{and} \qquad K_t(z,\zeta) = \frac{f^t{}_z(z)\, f^t{}_\zeta(\zeta)}{(f^t(z) - f^t(\zeta))^2}.$$

Then $$h_{l\bar{k}}(t) = \iint_{F(t)} \left[\frac{12}{\pi}\iint_H K(z,\zeta)^2\, \overline{v_k(\zeta)}\, d\xi d\eta\right] v_j(t)(z)\, dxdy$$

$$= \iint_F \left[\frac{12}{\pi}\iint_H K_t(\zeta,\bar{z})^2\, \overline{v_k(z)}\, v_j(\zeta)\, d\xi d\eta\right] dxdy.$$

Applying the fact that

$$K_t(\zeta,\bar{z}) = \frac{\partial 2}{\partial\zeta\partial\bar{z}} \log(f^t(\zeta) - f^t(\bar{z})) \qquad z, \zeta \in H$$

and some computation, we get

$$\frac{\partial K_t(\zeta,\bar{z})}{\partial t^l} = -\frac{1}{\pi}\iint_H \frac{f^t{}_\zeta(\zeta) - f^t{}_{\bar{z}}(\bar{z})}{(w - f^t(\zeta))^2(w - f^t(\bar{z}))^2} L^{\nu(t)}[v_l](w)\, dudv.$$

Then we have

$$\frac{\partial h_{j\bar{k}}}{\partial t^l}(t)$$

$$= \frac{24}{\pi}\iint_F \left[\iint_H \frac{\partial K_t(\zeta,\bar{z})}{\partial t^l} K_t(\zeta,\bar{z})\ \overline{v_k(z)}\ v_j(\zeta)\ d\xi d\eta\right] dxdy$$

$$= -\frac{24}{\pi^2}\iint_{F(t)}\left[\frac{12}{\pi}\iint_H K(\zeta,\bar{z})\ T_l(z,\zeta)\ \overline{L^{v(t)}[v_k](z)}\ L^{v(t)}[v_j](\zeta)\ d\xi d\eta\right] dxdy \quad (*)$$

where $T_l(z,\zeta) = \iint_H K(w,\bar{z})K(w,\zeta)\ L^{v(t)}[v_l](w)\ dudv$.

It can be checked that (*) converges uniformly and absolutely with respect to t. Then we can have

$$\frac{\partial h_{j\bar{k}}}{\partial t^l}(t)$$

$$= -\frac{24}{\pi^2}\iint_{F(t)}\left[\frac{12}{\pi}\iint_H K(w,\bar{z})\ T_j(z,w)\ \overline{L^{v(t)}[v_k](z)}\ L^{v(t)}[v_l](w)\ dudv\right] dxdy$$

$$= \frac{\partial h_{l\bar{k}}}{\partial t^j}(t).$$

□

A Differential Geometrical Interpretation of the Weil-Petersson Metric

Since $T(R) \cong \mathcal{M}(R)/\mathrm{Diff}_0(R)$, a metric on $T(R)$ induced by an inner product on $\mathcal{M}(R)$ gives the Weil-Petersson metric on $T(R)$.

Take a Riemannian metric $ds^2 = \rho^2|dz|^2$ on R which determines the conformal structure of R . ρ is a positive smooth function on U where (U,z) is a local coordinate on R.

The tangent space $\mathcal{T} = T_{ds^2}(M(R))$ consists of tensors of degree 2 on R. Every element α of $\mathcal{T}$ is written as

$$\alpha = Adzd\bar{z} + Bdz^2 + \bar{B}d\bar{z}^2$$

where A is a real-valued and B is a complex-valued smooth function on U. α

corresponds to a real symmetric matrix

$$\tilde{\alpha} = \frac{1}{2}\begin{bmatrix} A+B+\bar{B} & i(B-\bar{B}) \\ i(B-\bar{B}) & A-B-\bar{B} \end{bmatrix}.$$

Definition. The inner product of α_1 and α_2 in $\mathcal{T}$ is defined by

$$<\alpha_1,\alpha_2>_R = \iint_R \mathrm{tr}(\tilde{\alpha}_1\tilde{\alpha}_2)\rho^2\, dxdy$$

$$= \frac{1}{2}\iint_R \frac{A_1A_2 + 2(B_1\bar{B}_2 + \bar{B}_1B_2)}{\rho^2}\, dxdy\ .$$

The following 2 types of elements in $\mathcal{T}$ correspond to zero vectors in $T_0(T(R))$:

1. A vector induced by deformation of ρ. This is an infinitesimal deformation $\psi\rho^2|dz|^2$ which is generated by a 1-parameter family $\{\rho^2e^{t\psi}|dz|^2\}_{t\in\mathbf{R}}$ of conformal deformations of ds^2, where ψ is a real-valued function on R.

2. A vector induced by diffeomorphisms of R to R. Namely, this is a infinitesimal deformation

$$\rho^2\left(\frac{\partial\bar{a}}{\partial z}dz^2 + \frac{\partial a}{\partial\bar{z}}\, d\bar{z}^2\right)$$

induced by $\{f_t^*(ds^2)\}_{t\in\mathbf{R}}$ of deformations of ds^2, where $\{f_t\}_{t\in\mathbf{R}}$ is a 1-parameter family of transformations of R which is generated by a vector field $X = a(z)\frac{\partial}{\partial z}$ on R.

We want to obtain a condition on α such that α is orthogonal to all elements of type 1 and 2 in $\mathcal{T}$ with respect to $<\ ,\ >_R$. Firstly, in order that α satisfies

$$<\alpha,\beta>_R = \frac{1}{2}\iint_R A\psi\, dxdy = 0 \qquad \forall\ \beta \text{ of type 1.}$$

Then A = 0.

Secondly, in order that α satisfies

$$<\alpha,\beta>_R = \frac{1}{2}\iint_R \left(B\frac{\partial a}{\partial \bar{z}} + \bar{B}\frac{\partial \bar{a}}{\partial z} \right) dxdy = 0 \qquad \forall\ \beta \text{ of type 2.}$$

From Green's formula, we have

$$\iint_R \left(a\frac{\partial B}{\partial \bar{z}} + \bar{a}\frac{\partial \bar{B}}{\partial z} \right) dxdy = 0.$$

Since $X = a(z)\frac{\partial}{\partial z}$ is arbitrary, we have $\frac{\partial B}{\partial \bar{z}} = 0$. Then $Bdz^2 \in A_2(R)$.

As a result, α in $\mathcal{T}$ is orthogonal to all elements of type 1 and 2 if and only if $\alpha = \psi + \bar{\psi}$ for some $\psi \in A_2(R)$. Therefore, the subspace $\mathcal{T}_0 = \{ \psi + \bar{\psi} : \psi \in A_2(R) \}$ of $\mathcal{T}$ corresponds to $T_0(T(R))$.

The inner product of $\alpha_j = \psi_j + \bar{\psi}_j \in \mathcal{T}_0$ (j=1,2) is given by

$$<\alpha_1,\alpha_2> = 2\,\mathrm{Re}\iint_R \frac{\psi_1\ \overline{\psi_2}}{\rho^2}\, dxdy\ .$$

In particular, taking the hyperbolic metric on R as ds^2, the inner product coincides with the one on $T_0(T(R))$ which gives g_{WP} on T(R).

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